SPECIAL METHOD IN ARITHMETIC

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BY

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PREFACE

It is our purpose in this volume to make plain to teachers in elementary schools the purpose of teaching arithmetic, to outline fully a course of study based upon this controlling idea, and to discuss and illustrate the method of handling some of the chief topics.

It is strictly a teacher's book, and not designed for use by children.

It has been generally felt that arithmetic stands aloof from other studies, but the attempt is made here to show a very intimate and important relation to the other studies, one vital in fact to the proper plan and treatment of the whole subject.

This volume is one in a series of special methods, as follows: Reading of English Classics, Primary Reading and Story Work, Geography, History, Elementary Science, Language, and Manual Arts.

The course of study for the eight grades based on the plans worked out in these eight volumes is now in press and will appear in two volumes.

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SPECIAL METHOD IN ARITHMETIC

CHAPTER I

Brief Historical Review

The history of arithmetic as a subject of common school study and the widely varying opinions of its importance that have prevailed at different times and in different countries constitute an interesting chapter in education. It is only within comparatively recent times that arithmetic has been held of much importance. It had no recognition in the classical schools (like that of Sturm at Strassburg) established by the humanists during the Reformation. Nor among the early classical schools of England and America had it much standing. "It was toward the close of the eighteenth century that the modern treatment of elementary arithmetic began to show itself. In the Philanthropin at Dessau, an institution to which education owes not a little, we find in 1776 very little improvement upon the old plan of

pretending to teach all of counting, then all of addition, then all of subtraction, and so on. But in the following year Christian Trapp began upon entirely new lines, and in 1780 he published his 'Versuch einer Pädagogik,' in which he worked out quite a scheme of teaching young children how to add and subtract, objects being employed, and the effort being made to teach numbers rather than figures. This he followed by simple work in multiplication and division, and he worked out a systematic use of a box of blocks illustrating the relation of tens to units, a forerunner of the Tillich reckoning-chest mentioned later. It is here that we may say, with fair approximation to justice, the modern teaching of elementary arithmetic begins.

"After the adoption of the Arabic (Hindu) numerals in Europe, about 1500, arithmetic was taught for three hundred years in an extremely formal and abstract manner. It consisted mainly in figuring by rule, without effort to develop or explain the meaning of processes." (Professor David Eugene Smith, "The Teaching of Elementary Mathematics.")

In the first quarter of the nineteenth century Pestalozzi gave the strong impulse that led to the adoption of arithmetic as an important study in the

schools. His own school was famous for its success in number, and the methods he employed were followed and developed by his disciples, till finally Grube gave them a simple and systematic form which has been adopted largely in the schools of both Europe and America.

Pestalozzi gave a strong emphasis to *perception*, to the use of objects in developing the number sense. His successors worked out elaborate and systematic plans for illustrating number. The number ideas were developed before the figures were presented. Pestalozzi also gave great prominence to *oral work*, and in this connection to the vigorous *mental gymnastics* of arithmetic.

Number work was begun with children entering school in the first grade, and was continued through the course.

It would be interesting to trace the development of number teaching as worked out by Pestalozzi's disciples, by Tillich and Türk, Kranckes and Grube, because they show the extremes to which good ideas can be carried, and the reactions that naturally follow; for example, excessive emphasis of illustrative work, of mental gymnastic, of thoroughness, of logical order and mechanism.

These developments of method and of the course of study from Pestalozzi to Grube have had a powerful influence in shaping the present course of study in arithmetic, both in Germany and in the United States. An examination of the course of study worked out by Dr. Rein in his "Eight School Years" will show that he has adopted the Grube plan in a modified form, and has introduced into the method of treatment certain leading ideas of the Herbartian pedagogy, as follows: —

The old idea of beginning number work with observation is modified by Rein to the idea of starting out from things, and first of all not from artificial things such as cubes, blocks, abacus, splints, and other number devices, but from those familiar things and persons which children have known in their games at home, in nature study, and in the stories they have heard.

"The first number work should spring from that familiar group of objects already known to the children in story work and in nature study. . . . These objects and especially those common to the home are the very ones which press strongly for numerical expression and are the best basis for interest in number." When, through these familiar objects, the

interest in number has been once awakened, it can be carried over successfully to the splints, blocks, and number machines and systematic schoolroom devices. "When gradually the attention has been turned to the formal relations of number, the objects fall into the background, the number machine (abacus or blocks) assumes the observational phase or instruction, until even this support is unnecessary, and only the pure number remains. Thus we pass over from object to symbol, from symbol to the number idea, and in this manner safely transfer the interest from objects to the region of pure number." (Rein, Pickel, and Scheller; "First School Year.")

In building up the number idea in closest relation to home objects, to nature study, literature, and story, it is manifest that Rein has applied the doctrine of correlation vigorously to number work.

In the second place, in harmony with Herbartian pedagogy, Rein has reduced arithmetic to a relatively small number of large, and as far as possible, interesting, method units. The number six, for example, with all its possible combinations in the four processes, is one unit of thought. Its treatment should be based upon familiar experience, as, for example, the notion that the child in the first

year of school is six years old. The single combinations, as $4 + 2 = 6$, are not separately memorized at first but worked out in connected series (addition series, subtraction series, etc.). "Apart from the greatest possible objectivity in early number work, the chief emphasis is placed upon the working out of close and compact number series. The children should be induced and accustomed to build up these series without much questioning from the teacher. When once the series is firmly established, the following drill exercises in irregular order are in place." (Rein.) Here again it is clear that the working out of large units of thought, by building up series, by drills, comparisons, and applications, is based upon the formal steps of instruction.

Turning our attention to the history of arithmetic as a school subject in the United States, we observe first that for many years it has held a commanding place in the common schools. This has been due to two prevailing convictions: first, the notion that it was of great practical value to the common man; and second, the vigorous mental training believed to belong to the processes and solutions of arithmetic.

It is only within the last fifteen or twenty years, since 1885, that these standard articles of the arithmetical creed have been seriously questioned. A closer examination has shown that the parts of arithmetic which are of direct service in common life constitute but a small proportion of the exercises in our arithmetics.

"For the ordinary purposes of non-technical daily life we need little of pure arithmetic beyond (1) counting, the knowledge of numbers, and their representation to billions (the English thousand millions); (2) addition and multiplication of integers, or decimal fractions with not more than three decimal places, and of simple common fractions; (3) subtraction of integers and decimal fractions; and (4) a little division. Of applied arithmetic we need to know (1) a few tables of denominate numbers; (2) the simpler problems in reduction of such numbers, as from pounds to ounces; (3) a slight amount concerning addition and multiplication of such numbers, as from pounds to ounces; (4) some simple numerical geometry, including the mensuration of rectangles and parallelepipeds; and (5) enough of percentage to compute a commercial discount and the simple interest on a note.

"What, then, should be expected of a child in the way of the utilities of arithmetic? (1) A good working knowledge of the fundamental processes set forth above; (2) accuracy and reasonable rapidity, subjects which will be discussed later in this work; and (3) a knowledge of the ordinary problems of daily life. Were arithmetic taught for utilities alone, all this could be accomplished in about a third of the time now given to the subject." (Professor Smith, "The Teaching of Elementary Mathematics.")

This opinion seems now to prevail among thoughtful teachers, that only a small part of the old arithmetic had direct practical value.

The doctrine of mental discipline, of logical exercise through arithmetic, being grounded in a once prevailing psychology, is still held by many. But in the last few years the changes in psychological theory and in the whole course of study have been so great as to shake even this stronghold of the old arithmetic.

The doctrine of formal discipline has been largely undermined by the leading schools of modern psychology, both in America and in Europe. The old idea of separate faculties of the mind devel-

oped by different studies, for example the logical faculty by arithmetic, has now little recognition among psychologists. The doctrine of apperception has become in the main a substitute for the old notion of formal mental discipline.

As a consequence we are no longer willing to spend a large amount of time in drilling upon any subject further than it supplies us with ideas that will be useful to us in all our future studies and life. We do not study subjects for mere discipline, regardless of the positive ideas with which they equip the mind against future emergencies.

To exclude merely disciplinary topics from arithmetic is to shut out a large part of the difficult subjects of the old text-books.

A second influence, of equal importance, has demanded a curtailment of all doubtful topics in arithmetic. The common school course has expanded itself so vigorously of late into new fields that the demand has become imperative for a radical reduction and simplification of our whole curriculum. It would be preposterous to allow arithmetic to absorb so much school time as formerly, when such subjects as American history, nature study, choice literature, manual arts, physi-

cal training, music, and English are demanding, and properly, a full share of the child's time.

In the last few years there has been an elimination from the books of a large number of the obsolete and over difficult topics in arithmetic for grammar grades, such as partial payments, arbitration of exchange, and compound proportion. Just at the present time there is a strong demand for a still greater reduction, for dropping out much of common fractions, troy weight, true discount, partnership, cube root, equation of payments, and several others. There is an equally strong demand for an omission of difficult problems and greater concentration upon the simple elements of arithmetic so as to secure complete mastery of elementary processes. Arithmetic no longer stands by itself as an isolated instrument of training. The fundamental question is how can arithmetic work in coöperation with all the studies of the school to build up a well-organized and coherent body of knowledge such as a child needs.

It would not be difficult to show that all the leading ideas worked out by Pestalozzi and his successors have found more or less adoption in American schools. Our primary number work in graded

schools has been strongly modelled on the work of Pestalozzi and Grube. But in this country there has been of late a strong criticism of the Grube method by Dr. Dewey and others. Dr. Dewey bases the development of the number idea upon measurement, and measurement implies activity, the adjustment of means to end. The number one is not a fixed thing, but a standard unit with which to measure some larger, as yet undefined, whole.

"These two conceptions—(*a*) the origin of quantitative ideas in the process of valuation (measuring), and (*b*) the dependence of valuation upon the adjustment of means to end (*i.e.* ultimately upon activity) are the beginning of all conceptions of quantity and number, and the sound basis of all dealing with them." (McLellan and Dewey, "Psychology of Number," p. 41.)

This appeal to activity in measuring by means of the standard units, we believe to be a sounder basis for the number concept than mere observation as practised by Pestalozzi.

The limitation of the first year's arithmetic to the number space from 1 to 10, according to Grube, is also strongly criticised on several grounds: (1) that it does not correspond to the natural range of a

child's experience and interests; (2) that it sets up a standard of thoroughness wholly premature; and (3) that it involves a mechanical routine without proper motive or interest.

Notwithstanding this criticism, the first year's regular work in number (in our course, the second grade) includes most of the number combinations from 1 to 10, also the additions and subtractions from 10 to 20, and some exercises in still larger numbers, according to the natural scope of the children's knowledge and interests.

In this country, for several years, efforts have been made by teachers of various schools and tendencies to secure an intimate connection between arithmetic and other studies, to use arithmetic as a positive means of interpreting clearly many parts of a child's growing knowledge and experience. In this way arithmetic contributes directly to a better understanding by a child of his whole environment.

Even such a brief account as we have just given of the historical phases of number work suggests that a careful study of the various movements in number teaching is an excellent training in method. Historically all the different extreme conceptions of number work are strikingly exhibited by enthusiastic

advocates. There is also a steady movement to a broader and more satisfactory view of number in its nature and purpose in education. "The history of the subject gives us a point of view from which we can see with clearer vision the relative importance of the various subjects, what is absolute in the science and what the future is likely to demand. . . . So one who considers the historical development of arithmetic and its teaching will see how enormous has been the waste of time and energy, how useless has been much of the journey, and how certain chapters have crept in when they were important, and remained long after they became relatively useless." (Professor Smith, "The Teaching of Elementary Mathematics," p. 43.)

At the present time, with the light thrown upon our task by the history of method, we ought to gather up the controlling ideas that must organize for us our course of study in arithmetic.

1. Number work must spring in the lower grades from the activity of children in measuring and constructing with common standard units.

2. The clarification of number concepts can be greatly aided by the use of suitable, more or less artificial, illustrative materials, as the cubes and

blocks, the abacus and the splints for showing the decimal scale, etc.

3. Number work has its roots also in the consideration of those familiar objects and experiences which a child has encountered in his home, in reading and story, and in his other school studies.

4. The main units of study in the advance movement should be vague, quantitive wholes, requiring to be estimated and measured, thus giving motive to the work.

5. As we move into the work of intermediate and grammar grades, many topics from geography, manual arts, history, and science should be worked out quantitively, so as to give the number interpretation to otherwise vague and undefined topics. This implies that arithmetic can throw a great deal of light upon many important topics in other studies and find its chief utility, not in its immediate business applications, but in the broad illumination it throws over the whole field of knowledge and experience.

6. The whole trend of thought in the more recent discussion of arithmetic is toward greater thoroughness and mastery of elementary processes. This implies abundant oral work, vigorous and strong

mental drills, self-reliant thought and expression. In other words, the arithmetic must be strong and virile, toning up the mind to concentrated effort and to the formation of the best mental habits.

7. The application of the doctrine of the formal steps to arithmetic by Ziller and Rein, — by which the subject-matter of number is broken up into large and valuable units of study, which are worked out inductively from measurement and construction with objects to abstract number formula and process, — this use of the inductive-deductive method brings arithmetic under the operation of the same principles as are known to prevail in the other studies.

In this respect the formal steps are a means of organizing, into a consistent plan, the various principles of method which have been worked out at successive periods historically. This general method plan may be then rationally applied to those topics which in the sifting-out process of educational history remain as substantial elements of the arithmetical course.

CHAPTER II

AIM AND SCOPE OF ARITHMETIC

THE chief aim of arithmetic is the mastery of the world on the quantitive side through number concepts. This means the ability to estimate quantitively in numerical terms the varied objects and forces in the physical world as related to man.

To gain this mastery a child must have a varied number experience, especially in measuring the things of the objective world. He must measure, analyze, compare, and express the results in number language. He must become familiar with the standard measuring units (conventionalized and natural) which are in common use among people. Such are the yardstick, the day, hour, and year, the pound and ton, the dozen and score, the square foot and square mile, the gallon and bushel, the scale of degrees on the thermometer, the degree of longitude, the acre, the horse-power, the dollar. The use of any standard unit is made more complex by its relation to smaller and larger units of the same

scale, for which it must be often exchanged (as foot, with inch and yard). These measurements call for number expression and a great variety of number operations. As a consequence, we have a whole system of mathematical language with which the pupil must become quite familiar. It has taken thousands of years to work out a concise and satisfactory number language, a set of symbols of number and of number operations, simple and comprehensive, which now constitute an inheritance invaluable to every child. In measuring and comparing objects, the successive processes of addition, subtraction, multiplication, and division spring up in a natural order, and these in turn are followed by factoring, fractions and decimals, percentage, and involution, a whole series of processes which must become familiar to any one who wishes to express himself in mathematical language, or to understand others when they so express themselves.

Alongside these measurements and processes is a large number of positive number facts, which each child must fully memorize in order to make any progress in the mastery of the processes themselves.

While number itself is an abstraction, a mental act

by which we put number ideas into things, the presence of objects is still necessary in order to put the mind into an attitude for thinking number. In this sense we may say that number concepts begin with the measurement of physical realities and develop toward abstract number and numerical processes, but that in the application of number we constantly return to physical things, and are ever constructing and estimating our material surroundings numerically.

On the one side we may say that the purpose of number work is to put a child in possession of the machinery of calculation; on the other side it is to give him a better mastery of the world through a clear (mathematical) insight into the varied physical objects and activities. The whole world, from one point of view, can be definitely interpreted and appreciated by mathematical measurements and estimates. Arithmetic in the common school should give a child this point of view, the ability to see and estimate things with a mathematical eye. Whatever facts and processes are necessary to secure this kind of an eye and brain, the school should enable him to master.

The thing we aim at, therefore, is a completely practical and accurate mastery of our material sur-

roundings from the narrow point of view of number. It is not mathematical processes and discipline for their own sake. It is these as necessary instruments for achieving a definite result, the power to understand and make use of the world mathematically, the complete familiarity with the common standards as instruments applied definitely to all kinds of magnitudes. Moreover, the system of things that a child has occasion to know and interpret is the natural world as man has modified it; it is the world as seen from the standpoint of human conventions and institutions.

In this way it becomes evident that arithmetic stretches across all subjects, and includes a phase of all topics in the school course. It may sometimes seem to us that arithmetic is a wholly distinct branch of knowledge, and in its purely scientific and systematic form it is peculiarly abstract. It is not mathematics as a distinct and separate science that we have most to do with in the elementary school. It is rather the mathematical phase of every subject in the school course. Just as language, though a separate study, is present vitally in every study, so arithmetic, though distinct, is omnipresent in all subjects. If number, like language, is not present

in every study, there is serious weakness and defect. For number is a point of view, a perspective, into which every subject must fall, else the student is blind and ignorant on that side.

This aim of arithmetic, as a point of view to be gained by which we can more clearly interpret and subjugate all experience and all kinds of knowledge in every subject of study, will help us to solve a number of controverted points, *e.g.* What topics, once common and even now common, should be omitted from our text-books? How thorough shall be the mastery of processes? What kind of mental discipline shall arithmetic give? What kind of applied problems, and how difficult, are needed? What is the relation of arithmetic to other studies, and to a child's outside experiences? A clearly defined aim, such as we have proposed, should give a clear and positive answer to such questions.

Take the first question. What topics should be omitted from our text-books? Those whose facts and processes are not necessary as a general means of interpreting the child's world, *e.g.* they cannot use profitably the Roman or Chinese money standards, nor even the metric system in any complete form. All topics that are not broadly universal in

their application can be left to the technical specialist. Surveyors' measure, for example, is narrowly technical and special. It is not a mode of measurement in common use among any except professional surveyors. Omit also topics which are becoming obsolete because better ideas and processes are taking their place.

"On the basis of social custom, our brief standard, the following things — very often taught — may well be omitted: —

"Apothecaries' weight.

"Troy weight.

"Examples in longitude and time, except the very simplest, involving the 15° unit, since our standard time makes others unnecessary.

"The furlong in linear measure.

"The rood in square measure.

"The dram and the quarter in avoirdupois weight.

"The surveyor's table.

"Table on folding paper.

"All problems in reduction, ascending and descending, involving more than two steps.

"The G. C. D. as a separate topic, but not practice in detecting divisibility by 2, 3, 5, and 10.

"All work with L. C. M., except of such very common denominators as those just mentioned.

"Complex and compound fractions as separate topics.

"Compound proportion.

"Percentage as a separate topic, with its cases.

"True discount.

"Most problems in compound interest, and all in annual interest.

"Problems in partial payments, except those of a very simple kind.

"The same for commission and brokerage; for example, all problems involving fractions of shares.

"Profit and loss as a special topic.

"Equation of payments — made unnecessary by improved banking facilities.

"Partnership — made unnecessary, in the old sense, by stock companies.

"Cube root.

"All algebra, except such simple use of the equation as is directly helpful in arithmetic.

"In addition to all of these, arithmetic may be omitted as a separate study throughout the first year of school, on the ground that there is no need of it, if the number incidentally called for in other

work is properly attended to." (F. M. McMurry, "What Omissions are Advisable in the Present Course of Study, and what should be the Basis for the Same," address before National Department of Superintendence, 1904.)

The omission of the elaborate treatment of common fractions (which would save us at least a half-year's work) is proposed on the ground that the decimal fraction is far shorter, easier, and simpler. In common usage and in scientific work the decimal has already taken the place of the more difficult forms of the common fraction.

Another disputed point concerns the kind of discipline furnished by arithmetic. The old doctrine of discipline and thoroughness carried with it a large amount of pure drill and formal exercise. The idea of using number as measurement, as a means of interpreting better our surroundings, would limit number processes and drills to what is necessary to this controlling purpose. It would not spend time, for example, in writing numbers to ten periods, nor in memorizing and drilling on the squares of numbers from 13 to 25 and above.

Those phases of number study which are of value to every child, which help him to understand better

and quicker the varied objects of his experience, should be included in the number course. Those arithmetical processes which are least cumbersome and are most effective as ready means for necessary calculation, as the four fundamental processes, the decimal scale, and simple factoring, should be mastered. Those arithmetical processes, like the manipulation of complex and compound fractions. which are mere mental gymnastics, having lost their place in practical service, must be omitted. In this way we shall get a great reduction of time and mental strain and shall save energy for indispensable work in other fields.

Our controlling aim also suggests a much closer adjustment of arithmetic to the leading topics and investigations of other studies than has been usual in our manuals. Our text-books have used many so-called applied problems, but they have not made arithmetic, in any general way, a regular means of investigating and interpreting large topics in geography, history, and elementary science. This is now definitely proposed, so that the units of study in grammar grade arithmetic are typical occupations (as the profit and loss side of gardening or of managing a grocery store), administrative problems of

taxation and revenue (as the city finances), public enterprises and geographical problems (such as the Panama Canal). In short, we are prepared to turn arithmetical calculation directly into the main streams of other studies, to take up the large problems of history and industry and to work them out arithmetically. This does not mean that arithmetic loses itself in these studies or becomes absorbed into them so as to disappear as a distinct study. It means rather that in the grammar school course of study arithmetic should focus upon the same large and important units as the other studies. Such is the arithmetical treatment of irrigation in the arid states of the West, or the cotton production, sale, and shipment of the Southern states. These topics require an illumination from arithmetic which they can nowhere else get. To pass these topics through the alembic of mathematical calculation is to distil from them a new and stronger meaning. It is to get a point of view from which to estimate all practical human affairs more accurately and more simply. If arithmetic does not reach forward to this result, it does not clarify the general field of knowledge; it is abortive. It consists of a complex machinery for which we have not found the use.

Much has always been made of the practical uses of arithmetic, but it does not take long to show that these uses in the ordinary sense are very limited. The calculations employed in daily life are of the simplest character. On the other hand, the habit of putting arithmetical meaning into all the large, important problems of municipal and corporate enterprise, of raw production, commerce, and manufacture on a large scale, amounts to the possession of a strong vantage ground in education. The constant estimating and measuring of the great physical and national forces that man deals with is practical in the large educative sense, because it puts value and meaning into important things otherwise left vague and indefinite. Arithmetic is not merely a set of processes which one can master, and then later, as occasion arises, apply to various subjects. This view, if emphasized, gives a perverted notion of arithmetical development.

Arithmetic should grow and develop *pari passu* with other studies. Arithmetical thought should be built into the other studies steadily from day to day. Arithmetical facts and processes should be mastered just as fast as they can be used as instruments for interpreting and clarifying those new fields of knowl-

edge which are regularly coming into view. This is healthy and natural growth in power. This implies no waste in learning useless processes, nor a premature skill and speed in manipulating numbers for which we have no present use.

From this viewpoint arithmetic is a present light thrown brightly over nearly the whole field of study as rapidly as the topics of physical science, history, geography, and literature come into view. Without this light these studies would be lying partly in the dark, and arithmetic would be shedding its light upon no field of present interest.

The general scope of arithmetical study is determined, not by arithmetic alone, but also by the general range of school studies and child experiences, throughout which the sway of simple mathematical ideas should prevail. It is not too much to say that arithmetic leavens the whole lump.

Beyond this we are not anxious to build up any complete system of arithmetical thought or extraordinary skill in arithmetical reckoning. The working out and mastery of the science of number as a distinct body of thought, as a science, is not the aim of the common school course. But the same may be said of grammar and of elementary science.

The question as to the character of the mental discipline gained through arithmetic may be gauged by the same standard; that is, the ability to interpret promptly and mathematically the usual range of knowledge and experience. While this standard sets up a reasonable mastery of arithmetical processes, it runs to no extremes in mental gymnastic. It does greatly strengthen the motive for number work, and it gives a much richer field for self-activity and encouragement to the perpetual use of essential number processes. Anything which the school can do to encourage a more ready and flexible use of acquired knowledge in common affairs will give the best form of mental discipline. A reasonable degree of skill and power in number processes must be gained partly by reviews and sharp drills, but rigorous schoolmasters easily overstep this temperate limit and set up artificial tests of speed, accuracy, and power. These higher flights and exhibitions of dexterity and skill we can forego for the sake of that substantial and durable form of excellence whose distinguishing mark is aptness and inclination of mind to apply arithmetical standards and processes to every suitable field of knowledge.

The general culture value of arithmetic has been placed by some writers in its training to accuracy and precision of thought, in its discipline of the logical power, and in its exercise of the language faculty. Its value in these directions depends upon the vigor and skill with which it is taught. The general controlling principle which was laid down above for the whole course of study implies the proper inclusion of these elements of efficient teaching.

The general doctrine of the utility, the serviceableness, of arithmetic as an interpreter of all our surroundings from the number point of view rescues it from all narrow and sordid considerations. It is not utility in the bread-and-butter sense, but that broad catholicity of service which should distinguish all the common school studies. Each study opens up the whole broad field of knowledge, but views and interprets the whole from a distinctive and penetrating point of view. Arithmetic sweeps the whole expanse of a child's knowledge and experience with a peculiar and powerful searchlight, — that of number calculation. The result is the mastery of the world from that important point of view.

CHAPTER III

Method in Arithmetic

Primary Grades. First, Second, and Third Grades

In this chapter on method, it is my purpose to show how children, in pursuing the course of study laid out in the chapter following this, may appropriate its value in the most natural and economic way. The ideas suggested in the previous chapters will be definitely organized and applied. To simplify and concentrate the labors in arithmetic upon essentials, to avoid wasted effort, to give continuity and unity to the entire course of study, is the aim of the present chapter.

The problem of method in arithmetic, as in other studies, is the problem of mastering and applying principles and of combining them effectively into an organized whole.

But arithmetic as a science is extremely abstract. The first great question, how to concrete arithmetic and to relate it closely to a child's experience, has been a source of much controversy, and

has led both the theorists and practical teachers into widely divergent, and even contradictory, plans of study.

On the assumption that the number idea and its developments are based upon measurement, as demonstrated by Dewey and McLellan in their "Psychology of Number," we will first run through the varied series of concrete materials in second and third grades which may be employed as the basis for activities in measuring and which also furnish an opportunity for the number idea to develop.

"Number arises in and through the activity of mind in dealing with objects. . . . *Number* is not (psychologically) got *from* things, it is put *into* them. It is almost equally absurd to attempt to teach numerical ideas and processes *without* things and to teach them simply *by* things. Numerical ideas can be normally acquired and numerical operations fully mastered only by arrangement of things — that is, by certain acts of mental construction, which are aided, of course, by acts of physical construction; it is not the mere perception of the things which gives us the idea, but the *employing of the things in a constructive way.*

"In reality, it arises from the constructive (psychical) activity, from the actual use of certain things in reaching a certain end. This method of constructive use unites in itself the principles of both abstract reasoning and of definite sense observation." (Dewey and McLellan, pp. 60, 61, and 62.)

This thoughtful physical and mental activity in measuring, comparing, and relating objects or parts of a whole is the source from which number ideas spring.

There are several kinds of objective material that may be used as a basis for activity in measuring.

1. One of the first is the *counting* of physical objects, as chairs, birds, houses, children, apples, fence posts, etc. Children take to this easily before the school age, and it may be continued. It is a crude form of measuring with the eye, the children often pointing at or touching the objects. It soon passes over into mere counting to one hundred, in which they take pleasure.

2. Miscellaneous objects, such as toothpicks, apples, buttons, beads, schoolroom objects, pencils, fingers, may serve at any time to illustrate ideas of number. The teacher should have a stock of these things at hand, and should be fertile in find-

ing others at need. The defects of this class of materials are easily noted: (*a*) the units are often not of uniform size and quality; (*b*) they do not illustrate so well the idea of measurement; (*c*) and they do not furnish natural wholes to be analyzed and recombined.

3. A continuation of the administrative devices used during the first year in developing ideas of number, as the distribution of papers, books, and pencils to pupils, the marking off of positions at the board, the paging of books, counting time on the clock face, the attendance record of the school, games in and out of doors, sports (keeping score, jumping, and running). These involve measurement in more definite forms and are closely allied to interesting duties and activities of the children. In this connection we may also include the abundant and varied number work suggested by and incidental to other studies, as the constructions in drawing lessons, measurements for manual training work, the records of nature study, observations of weather (thermometers), of plants, and animals. In the third grade the beginnings of home geography suggest a multitude of interesting quantitive studies, as houses and building, map drawing, etc.

4. The abacus, the counting frame, and the measured cubes and blocks. These are artificial illustrative materials designed in particular for number work. The children themselves can handle the abacus and measure off the combinations, adding, subtracting, dividing, and multiplying. The blocks should include sixty inch-cubes of wood, ten parallelepiped blocks of each length (two-inch, three-inch, etc., to ten-inch). As later explained, these blocks will give relief from the single combinations of the Grube method. The blocks also serve later as an introduction to the computation of surfaces and areas, and as a preliminary to square and cubic measure, and to square and cube root.

5. The standard units of denominate numbers furnish the ideal measuring units. Such are the foot, yard, gallon, pound, dollar, bushel, barrel, day, dozen, cord, cubic foot, etc. The primary school should be supplied with sets of the more common and easily handled of these units, *e.g.* pint, quart, and gallon cans, peck and bushel measures, foot, yardstick, and tape-line, a pair of scales and set of weights, a clock face with movable hands, calendars, dollars, dimes, and pennies, or their imitations. The children in primary grades

should handle them freely in actual measurements of distances, heights, liquids, sawdust or sand, weights, and values.

There are several advantages in these standard units. (*a*) They give free and varied activity in accurate measuring and in perceiving of relations. (*b*) They are common standards used everywhere in this country, and will remain through life the foundation of all sorts of quantitive measurements of material things in all studies and in all experience. There will be nothing to unlearn nor to forget. (*c*) In relating these standard units to the lesser and larger units in the same scale or table, we secure those common ratios and fractions which will be of greatest service in all later arithmetical work, *e.g.* in pint, quart, and gallon, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{4}$, etc.; in cent and dime, $\frac{1}{10}$, $\frac{1}{5}$, etc.; in inch and foot, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{6}$, etc.; in ounce and pound, $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{8}{16}$, etc. These ratios include all the number combinations from 1 to 10, except those in 7 and 9. These tables bring out concretely also the relations of many numbers from 10 to 100 and 2000, including the series of the decimal scale in United States money.

6. In the second grade, number pictures expressed by dots or lines in proper arrangement

have served well the purpose of presenting number relation sharply through the eye, *e.g.* ⁙ and 5, $\begin{smallmatrix}\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\end{smallmatrix}$ and 9, etc. The use of dominoes and cards serves a similar purpose.

In learning to write numbers at the board or on paper children may use these number pictures and the symbols interchangeably, or as equivalents, with a much clearer perception of the meaning of the corresponding figures.

7. In teaching the decimal scale in third grade, splints or toothpicks, held in bundles of 10's and 100's by rubber bands, are extremely useful in the first introduction and explanation of the series, and the relation of the successive units to one another. Numbers like 23 and 46 can be added, or the difference found, also $825 + 143$. Later $36 + 78$ and $365 + 247$, by separating the bundles into lesser units and recombining them. These operations can be performed by either pupils or teacher until the fundamental idea of the decimal scale is clear. Even simple multiplication and division can be illustrated in this way. No long-continued use of the splints for this purpose is necessary, because it is desirable, when the notion of the scale is once

clear, to pass over to a quicker and less concrete handling of number relations and processes.

In all these seven modes of concreting elementary number operations there is great danger of overdoing the matter by continuing too long in objective work, thus converting it into a routine. In all cases it is necessary to push on to the more rapid and abstract treatment of number, always keeping the way open to a quick return to the concrete, when there is lack of clearness in thinking.

The next large problem is to determine the order of the systematic treatment of the number facts, and the proper distribution of the fundamental facts and operations through the primary grades (second and third grades). Here, again, we enter the field of widely divergent theory and practice.

The widespread adoption of the Grube method in this country and in Europe, and its familiarity to teachers make it a convenient starting-point for an inquiry into the true order of number studies.

According to the Grube method, the first year in number work (either first or second grade) is devoted to a thorough study and mastery of number combinations in the number space from one to ten. This includes all the number facts of addi-

tion, subtraction, multiplication, and division in regular order for each number simultaneously from 1 to 10.

This plan of number work has commended itself to the judgment of many of the best schoolmasters, (*a*) because of its simplicity both to teachers and pupils, (*b*) because of the idea of thoroughness, of absolute mastery of elementary facts, fundamental to all later arithmetical operations. It has also been worked out and applied with a complete system of objective, illustrative devices. In Germany, it was the final outcome of Pestalozzi's impulse toward concreting and planning number study. Arithmetic was Pestalozzi's favorite subject for illustrating his educational ideas. In Germany earlier, and in this country more recently, Grube's method has been subjected to a searching criticism, in America at the hands of Dr. Dewey and Professor David Eugene Smith, and largely modified on psychological grounds.

The objections to the Grube method on the psychological side, as stated by Dr. Dewey, are as follows: "The unit is never to be taught as a fixed thing (*e.g.* as in the Grube method), but always as a unit of measurement. One is never one thing simply, but always that one thing *used as a*

basis for counting off and thus measuring some whole or quantity" (p. 80).

"The method which neglects to recognize number as measurement (or definition of the numerical value of a given magnitude) and considers it simply as a plurality of fixed units, necessarily leads to exhausting and mechanical drill. The psychological account shows that the natural beginning of number is a whole needing measurement; the Grube method (with many methods in all but name identical with the Grube) says that some one thing is the natural beginning from which we proceed to two things, then to three things, and so on. Two, three, etc., being fixed, it becomes necessary to master each before going on to the next. Unless four is exhaustively mastered, five cannot be understood. The conclusion that six months or a year should be spent in studying numbers from 1 to 5, or from 1 to 10, the learner exhausting all the combinations in each lower number before proceeding to the higher, follows quite logically from the premises." ("Psychology of Number," p. 85.)

David Eugene Smith says, "A more tedious way of presenting number than that of Grube's would

be hard to find." ("The Teaching of Elementary Mathematics," p. 118.)

Our own experience confirms this judgment in part and convinces us that even with good teachers the strict Grube method is tiresome and monotonous, without proper scope and adaptation to a child's other motives and interests.

The second count against the Grube method by the psychologist is that the learning simultaneously of all the number relations, for example, in 5, before passing on to 6 (addition, subtraction, multiplication, and division), until complete mastery is attained, is unpsychological, because the different processes are not equally difficult. The additions and subtractions can be understood and memorized a considerable period before the *times* idea in multiplication and the ratio idea have developed into clearness. Dewey says, "It seems absurd or worse than absurd to insist on thoroughness, on perfect number concepts, at a time when perfection is impossible, and to ignore the conditions under which alone perfect concepts can arise — the wise working with imperfect ideas till in good time, under the law connecting idea and action, facile doing may result in perfect knowing." Again: "It is plain that there must be time for the develop-

ment of this abstracting and generalizing power. In fact, the complete development of the 'times' idea, this factor relation, corresponds with the stages of the measuring power of number. The higher power of numerical abstraction is the higher power of the tool of measurement. This normal growth in the power of abstracting and relating cannot be forced by any — the most minute and ingenious — analyses on the part of the teacher. The learner may indeed be drilled in such analyses, and may glibly repeat as well as 'reason out' the processes; just as he can be drilled to the repetition of the words of an unknown tongue, or any other product of mere sensuous association. But it does not follow that he knows number, that he has grasped the idea of times." ("Psychology of Number," p. 110.)

In the third place, to limit the child's number work for a whole year to the number space from 1 to 10 not only includes the mastery of operations which should be incidental and allowed to develop more slowly (multiplication and division); but it excludes a free range among higher numbers, some of which are easier to handle and more closely allied to a child's interests and activities than the combinations, for example, in 7 and 9.

Dewey says: "Instead of relying upon a minute and exhaustive drill in numbers from 1 to 5, allowing next to no spontaneity, severing nearly all connection with a child's actual experience, ruling out all variety as diametrically opposed to its method, it (the measuring method) can lay hold of and give free play to any and every interest in a whole which comes up in a child's life. Unity as 12, as a dozen, is likely to be indefinitely more familiar and interesting to a child than 7; the desire to be able to tell time comes to be an internal demand, etc. But the Grube method must rule out 12. Twenty-five as a unity (of money, the quarter-dollar), 50 (as the half-dollar), 100 (as the dollar), are continual and lively interests in the child's own activities. Each of these is just as much one as is one eye or one block, and is arithmetically a very much better type of the unit than the block by itself, because it is capable of definite measurement, or rhythmic analysis into sub-units, thus involving division, multiplication, fractions, etc., operations which are entirely external and irrelevant to the fixed unit." (McLellan and Dewey, p. 89.)

At the very beginning of a child's experience with number (in counting) he spontaneously breaks over the 10 limit and counts to 100. The very least

encouragement will induce a child of six to count by 1's to 100 and then by 2's, by 10's, and 5's to 100. On the other hand the objects which are the starting-points of his interests and activities are often expressed and recommended to him in the form of larger concrete numbers: the foot of 12 inches, the clock face, 12, the book with its 100 or more pages, the number of pupils in the school, 30, the price of a book, 50 cents, the cost of a pair of shoes, the frontage of a town lot, 60 or 100 feet, the figures on a thermometer or a pair of scales.

In scores of directions the child's number sense outruns the 10 limit before he has finished his first year in school. In spite of this, however, the lower numbers are of chief importance in the first number study.

Without knowing anything explicitly as yet about the decimal scale, children may deal with tens, hundreds, and even thousands as representing familiar and oft-recurring experiences, *e.g.* the price of strawberries in the market (10 or 20 cents), prices of horses or cows ($60, $150, etc.), weight of coal in tons (2000 lb.), the price of town lots ($400, $800), etc. To shut these things out is to disregard the natural range of a child's experiences and interests. It is

desirable and practicable, however, to throw the chief stress in the second year upon the number space from 1 to 20.

From the standpoint of method the first great question to be asked in planning number work is: What are the natural and appropriate *units of study* for children? What are the important objects confronting the child which call for measurement and mathematical interpretation? This is a very different question from that other question, asked by the confident schoolmaster: What are the simple and elementary number combinations which every child will need to know and which will be necessary for all his later studies? The first question is one vital to method, the second may be the basis of a dry routine.

In the main the units of study which should serve as the starting-points and ground of interest in measurement and numerical calculation should be vague but interesting wholes, which confront the child in his ordinary experience and naturally call for analysis and numerical definition. How many seeds are there in a pod? How much do the children weigh, or how tall are they? How large is the schoolroom, or dining room? How far do you

live from the post-office, in blocks? What is the cost of a doll? How large shall we make a garden bed? How many boxes of berries in a crate? How many books in the school library? How much will it cost to get a pony, and how much feed does he require? The price of a piano, of a suit of clothes, or of a book, the ages of school children, the number of months in the year or of hours in the day, the number of pupils required to play a game, the score, a bunch of fire-crackers, the size of a pail or basket or barrel,—these are a few of many things calling for measurement, and the standard units are easily at hand with which to find the answers.

"*Work from and within a whole.*—Here, as everywhere, the idea of a magnitude—a whole quantity—corresponding to some one unified activity should be present from the first. Some vague quantity or whole, which is to be measured by the putting together of a number of parts, alone gives any reason for performing the operation and sets any limit to it. The process of breaking up the whole into parts, and then putting together these parts into a whole, measures or defines what was originally a vague magnitude and gives it

precise numerical value." (McLellan and Dewey, p. 105.)

"It may be laid down, then, in the most emphatic terms, that the value of any device for teaching addition depends upon whether or not it *begins with a whole which may be intuitively presented*, and whether or not it proceeds by the rhythmic partition of this original whole into minor wholes, and their recombination." (McLellan and Dewey, p. 107.)

The Grube method is a simple routine of memory drills, relieved somewhat by illustrative devices. It is too simple for the child. There is not enough in it to engage his full attention. It lacks motive and sets up the standard of absolute perfection too early. This must be said in the very teeth of the schoolmaster's demand that a child shall know something and shall know it thoroughly. Drill and routine we must have, but they should follow in the wake of well-motived work, of rational units of study.

Arithmetic is passing through a stage of development very similar to that experienced a little earlier in the teaching of reading. Not long ago it was customary in reading to begin with alphabets and sounds and have the children work their way through these fundamental elements to words and sentences and

eventually to stories (a purely synthetic routine). We now begin reading with large units of commanding interest, as a story, picture, or plant, and by a combination of analytic and synthetic methods gain a mastery of reading. Thoroughness in the fundamental elements (names of letters and sounds) is now merely an incidental part of a thought movement originating in and sustained by central units of interesting study.

The Grube method is a perfect illustration in arithmetic of what we used to do in reading. It is purely synthetic, absolutely invariable in its movement from the simplest elements to larger combinations, and to a large extent regardless of the natural and interesting units of study which spring up about the child. Many of these study units lie within the number space from 1 to 10, as a child's age, the number of fingers, a gallon measure, the price of an orange. Others lie beyond this. It is necessary, of course, to concentrate study as much as can be reasonably upon simple numbers.

Even in combining and separating numbers from 1 to 10 there is much opportunity for building up and tearing down the larger units, forming thus continuous *series*, which give a movement of thought.

For example, in building up the number 7 with cubic inches and blocks (parallelepipeds of different lengths), we have the following series: —

$6+1=7$	$3+4=7$	$2+2+2+1=7$
$5+2=7$	$2+5=7$	$3+3+1=7$
$4+3=7$	$1+6=7$	

Similar series can be built up and taken down with all the number units to 10. Constructing such series with measured blocks (either addition or subtraction, multiplication or division) demands not only physical activity in measuring, but continuous attention and thought movement in order to keep track of the series.

In following the Grube method it is not unusual to find the teacher devoting a period of twenty minutes in drilling upon $3+4=7$, 3 apples + 4 apples = 7 apples, 3 toothpicks + 4 toothpicks = 7 toothpicks. A boy pays 3 cents for apples and 4 cents for a pencil, in all he spent 7 cents, etc., etc. In this way by sheer repetition the ball is kept rolling. But there is no conceivable device or group of devices by which the attention can be kept stationary for twenty minutes upon such a small point. Attention does not act in that way. There must be an object of

interest and a movement of thought in order to maintain attention, and this the series allows and requires.

It is not claimed that such series secure a complete mastery of the single additions, subtractions, etc., but they bring about a comparison of a closely related series of additions or subtractions and lead to a quicker and more thoughtful memory of the individual sums or differences.

It would be much easier at first to follow the simple, unchanging order of the Grube method than to search about in a child's life for the units of thought which command his curiosity and interest in his efforts to reduce these varied articles to mathematical measurement. But the Grube method is too narrow in its range and too invariable in its routine to allow of full consideration of a child's interests, motives, or range of experiences.

Our assumption is that number stimuli are everywhere appearing in a child's life, in his games and home affairs, in his other studies, in all his observations and experiences. It is not merely a school hour's routine, perfect in its isolation, but an attitude of mind toward all things, a mode of approach and interpretation for the whole environment. To

such an extent is it true that the number idea is involved in all a child's doings, that some of the best teachers are willing to leave out number work as an independent study from the first school year, or even the second, on the ground that the children in all their work and play are picking up number experiences, and that it is better for a while to let the number idea grow spontaneously, with only incidental recognition of it in the other studies.

Recognizing, therefore, the presence of this number interest in children and the countless opportunities to gratify it, the question arises how to select as lesson units a sufficient number of the most appropriate objects or series of things upon which to work out all the fundamental facts and processes of number.

One of the earliest number impulses is that of counting, — counting by 1's to 50, or to 100, later by 2's, 10's, and 5's. They use objects at first, and later seem to enjoy pure counting. But in school work the concrete background should be kept close at hand for reference and illustration. Counting is a crude form of measuring, and is built up at first upon numerous objects, such as trees, apples, cattle, houses, desks, etc. Counting is the simplest form of the series, giving a continuous mental move-

ment, and serving in its memory products as a basis for addition, subtraction, multiplication, and division.

In treating the number space between 1 and 10, using the standard units of denominate numbers, as gallon, yard, peck, bushel, dime, etc., also other materials, as the measured blocks, fingers, abacus, and number pictures, as a basis of measurement, we should first form with each number the addition and subtraction series, or let the children do so, then give the drills in irregular order to fix the individual sums and differences.

The series for 5 would stand: —

$1+4=5$	$2+3=5$	$3+2=5$	$4+1=5$
$5-1=4$	$5-2=3$	$5-3=2$	$5-4=1$

$$2+2+1=5$$

There should be no haste to understand and memorize the multiplication and division tables, yet they are implied in the preceding.

"It is a mistake, for various reasons, to attempt to treat the four fundamental processes simultaneously. They are not of equal difficulty; the child does not need them to an equal degree; and the world of business does not use them with equal frequency. Hence addition, the easiest, the most important, and the

most interesting to the child, occupies the chief attention in this year. Incidentally, as needed in the simple problems proposed, the ideas of subtraction, multiplication, and division are introduced; but the work, even in addition, is limited to the number space from 1 to 20, and no tables are learned." (Smith and McMurry, "Mathematics in the Elementary School.")

"*Simultaneous Method not Psychological.* — It seems clear, therefore, that the fundamental operations as formal processes should not be taught together; on the other hand, rational *use* should be made of their logical and psychological correlation. It is one thing to perform arithmetical operations in such a way as to *involve* the *use* of correlative operations, and it is another thing to force these operations into consciousness, or to make them the express object of attention. The natural psychological law, in all cases, is first the use of the process in a rational way, and then, after it has become familiar, abstract recognition of it.

"Then, the common method errs in the opposite extreme by attempting to force the recognition of ratio and fractions into consciousness before the mind is sufficiently mature, or sufficiently exercised in the

use of ratio, to grasp its meaning. The result of this unnatural method is that mechanical drill and memorizing, with the sure effect of waning interest and feeble thought, is forced upon the pupil. To master all the numerical operations contained in 6, 7, 8, and 9 is a slow and tedious process, and so the method is compelled in self-consistency to limit the range of numbers which are to be mastered in a given time. In reality, it is easy for the mind to grasp the fact that $1 is a hundred 1's, or fifty 2's, or ten 10's, or five 20's, long before it has exhausted all possible operations with such numbers as 7 or 11 or 18." ("The Psychology of Number," pp. 100, 101.)

In learning the multiplication tables in third grade, the addition series, by 2's, 10's, and 5's, can be reviewed, then the counting by 3's, 4's, etc., to 9's, follows as a preparation for the tables. In learning the multiplication tables there is a good chance to review the addition tables, to bring out the ratio idea, and the review of simple fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{2}{3}$, etc. After a free use of addition and subtraction, the idea of repeating similar addends a number of *times* will spring up, and the shortening of the process by multiplication is then a rational relief.

The multiplication table may be learned in a much easier order than is usual, as follows: 10's, 2's, 5's, 4's, 8's, 3's, 6's, 9's, 7's. Advantage should be taken of the similarity between the 2's, 4's, and 8's, also between the 3's, 6's, and 9's. Note also the fact that in counting by 8 the right-hand figure expresses 2 less each time, in counting by 9's one less. Children are interested and curious about these things, and they aid the memory. The number 24 equals 4×6, 3×8, and 2×12. Such cases should be examined, and reasons given for this variety of factors in the same product.

In learning the multiplication tables later, the memory can be aided by a variety of these reviews, comparisons, and rational analyses.

First of all, the additions based upon objective work and measurement stand in the background of thought, as giving meaning to all multiplications and divisions. Secondly, the repetitions and regularities running through some of the tables should be studied as of curious interest, as in case of the 10's and 5's, 4's and 8's, etc. Third, the identity of certain products in different tables, as $4 \times 5 = 20$, and $5 \times 4 = 20$, and $2 \times 10 = 20$. A comparison and analysis of these identities is excellent thought work and a positive

aid to the memory. It opens up the question of factoring, and the commutative law which will be so important later (3×5 ft. $= 15$ ft., the same as 5×3 ft. $= 15$ ft.).

If the 7's are placed last, as the most difficult table, it may be discovered that there is nothing new in it. All its combinations have appeared in the previous tables, on the basis of the commutative law, $7 \times 5 = 5 \times 7$.

The fact that the multiplication table has been learned usually in regular order, from 1 to 10, without regard to the proper order as to ease of acquisition, and that there has been so little regard for the rational connections and surprising uniformities within the whole table, is a remarkable proof of the thoughtlessness of teachers, and of the overwhelming natural tendency to drop into routine methods.

In teaching the decimal scale in third grade the use of splints or toothpicks wrapped in bundles of tens and hundreds is a good mode of objectifying the decimal series. By placing a table with the splints properly arranged before the class, it is possible to illustrate the steps in the scale, the correspondence of the numbers to objects, as in

the number 238, the addition and subtraction, with the breaking up of the bundles to permit of addition or subtraction, and even multiplication and division.

The children also should be allowed to illustrate these processes with the splints. Later, cents, dimes, and dollars may be used to show similar processes, though not so completely.

In primary grades the great bulk of the number exercises should be oral, without the use of pencil or blackboard. Even when the problems are assigned in the book, most of them can be best worked out mentally, that is, the operations performed without a pencil.

A few minutes each day spent in rapid reviews and drills in the four fundamental operations will do more than anything else to attain accuracy and speed. In such work it is easy for the teacher to command class attention so that every child is alert and attentive. Wherever any one is listless or inattentive, the questions should drop upon him especially and awaken effort. Naturally slow pupils, however, should be given time to think, and not be required to keep up a speed which only quicker pupils can attain. The slow ones often prove the

stronger and abler in the end. There is no subject in which pupils are so easily flustered or confused, and sapped of confidence in themselves. Kindly tact and consideration, therefore, are needed by many pupils in order to bring them to a confident use of their ability. In making up oral problems the teacher should keep within the children's experience of local objects and interests. An excellent test of the power of children to think is made by many teachers in requiring the children to make up problems, *i.e.* give number stories.

"If the oral work is rapid and accurate, the written work will be so. But in quantity the oral work should predominate. There is a temptation to have too much written arithmetic at this stage, simply because it is easily assigned for seat work. The great danger is that such young children, thus left alone at their seats, will drop into careless habits, involving division of attention. No text-book is advisable, at least in the first part of this year [second grade], for reasons previously given." (Smith and McMurry, "Mathematics in the Elementary Schools.")

"Revived by Pestalozzi and his contemporaries, the oral work had much favor, not only in Europe, but also, thanks to Colburn's excellent work, in

America. But the advent of cheap slates and paper and pencils seems to have driven it out of our schools for a generation. It is now reviving, and it is to be hoped that we shall not again cease to secure reasonable facility in rapid oral work with the ordinary numbers of daily life. The subject can easily be carried to an extreme; but within reasonable limits it should be demanded in every grade. It lubricates the arithmetical machine, and five minutes a day to this subject could hardly fail to bring all pupils to reasonable facility with numbers." (Smith, "The Teaching of Elementary Mathematics," pp. 117, 118.) In all the primary grades there should be considerable board work, by which the children become familiar with a free round hand in forming the digits, an easy acquaintance with the symbols of operation, and readiness in expressing all the simple operations. The freedom and readiness with which the teacher uses the board for illustrative diagram and problem will give the children a strong example and impulse to do written work.

Seat work with written problems should be short and not tediously anxious and exact. Without being careless and inexact on the one side, or over

exact and formal on the other, it can be free and orderly, and neat in appearance.

"Written work is still subordinate, being confined largely to computations too extensive for purely mental treatment. A large amount of rapid oral analysis should be given in very abbreviated form. Elaborate analysis should be avoided both by the teacher and the pupil.

"There is great advantage in the occasional use of oral analysis giving the main steps without performing the operations. For example: If a book costs $2, how much will 36 such books cost? Answer: '36 books will cost 36 times $2.'" (Smith and McMurry, p. 28.)

Even in the third grade the power to read and interpret simple practical problems should be definitely cultivated. It is based largely upon ability to image clearly the conditions of the problem. This can often be assisted by diagrams. For example: A garden is 3½ rd. wide and 20 rd. long; how many square rods in it? Draw the diagram and mark it off and let children learn to do it. The teacher will do well to invent many modes of diagram and representation so as to encourage the habit of clear thinking. Abstract rapid work will follow later.

CHAPTER IV

Method in Arithmetic (*Continued*)

Intermediate Classes. Fourth, Fifth, and Sixth Grades

In these grades we may enter upon a serious discussion of the problem of teaching arithmetical processes and their application. The simpler processes of arithmetical computation have already become familiar in primary arithmetic. But it is in the intermediate grades that special prominence is given to the mastery of these processes.

These are capable of being stated in the form of definitions and rules which, however, we are in no haste to formulate; in other words, they are general forms of procedure which can be illustrated by concrete examples.

Accordingly the study of arithmetical processes furnishes one of the best opportunities to apply inductive methods. Each process is uniform, and under each are thousands of examples from which a few may be chosen with which to introduce and

illustrate it. This is true, for example, of the addition of fractions, of percentage, of decimal notation, etc. When once worked out, these processes are found to apply in a many-sided and far-reaching way to a very wide range of common human experiences. By applying mathematical measurements and processes we clarify, simplify, and bring into order a multitude of business relations, of social and economic interests, of historical, geographical, and scientific topics, which without this illumination from arithmetic would remain hazy and mythical.

Nearly every topic in arithmetic has these two phases: first, to derive these general processes; second, to apply them variously to important practical and theoretic affairs that need arithmetical clarification.

Arithmetic is one of the best studies we have to illustrate the complete inductive-deductive thought movement.

In addition to this we may observe that the various processes in arithmetic have a very close kinship and dependence upon one another, making it necessary constantly to remember and utilize earlier processes in later work. This gives arithmetic a peculiar strength and value as a logical and developing

subject. It compels a constant enlargement and organization of the older topics as they reach into and interpret later processes and more complex practical problems.

Arithmetical processes and their modes of expression are the means by which we give quantitive interpretation to the large masses of things and to the active forces that we see in the physical world. We are constantly measuring, comparing, and simplifying to ourselves these at first vague and unestimated agencies about us. The units of thought, therefore, in arithmetic are sometimes these number processes, and sometimes they are the practical life problems (in commerce, in the household, in business) which only number processes enable us to work out to a solution. The need of arithmetic in the common affairs of life and the close correlation of its processes with geographical, scientific, and economic questions justify the selection of many important units of study from the field of applied arithmetic.

How to handle these larger units of study (both processes and practical problems) is the chief question to be answered in method.

The plan of proceeding from particular illustrations (through induction) to a mastery of general processes

and their application (deduction) gives us the outline of method. This plan has been fully discussed and illustrated in the "Method of the Recitation," in its general application to all studies.

It remains to work out its more detailed application to the peculiar difficulties and lesson units of arithmetic.

The process of dividing in long division may serve to introduce the chief steps in this movement toward the mastery of a general process. The children have learned the process of short division so that they can use divisors from 1 to 10 easily; *e.g.* such problems as the following they can easily work. At six dollars an acre for rent, how many acres can be leased for $1524?

$$\begin{array}{r} \$6\overline{)\$1524} \\ \hline 254 \end{array}$$

They wish now to learn to divide with large divisors, such as 16, 36, and 428. They are at first totally in the dark as to how this is to be accomplished, and yet they can easily see that it may be very desirable to know how to work with large divisors. For example, a farmer might wish to know how many horses he could buy, at $76 each, for $912. We have, therefore, a genuine question, of some

interest to children, which may be set up as the aim to be reached. In laying out town lots a man might wish to divide a large tract into lots 80 ft. or 96 ft. wide at the front. It would be well in a case like this to make a few actual measurements in the village or city to see how lots are divided. After such experimental measuring, a few simple oral problems may be worked in which lots of 50 and 60, or 80 ft. frontage are made the basis of reckoning.

We might simply work out a common example in long division; as, 1428 divided by 21 :—

```
21)1428(68
   126
   ---
    168
    168
    ---
```

and show the method directly. A second and third could be worked out, and so on, till the child begins to imitate the steps of this process, without any insight into the reasons. This is not an unusual method.

A better plan is to recall the mode of dividing in short division and review the explanation. Show also that this can be expressed in the form of long division; *e.g.* :—

```
4)3684        4)3684(921
   921          36
                --
                  8
                  8
                  -
                   4
                   4
                   -
```

Explain as follows: One-fourth of 36 hundreds is 9 hundreds. One-fourth of 8 tens is 2 tens. One-fourth of 4 units is 1 unit. Summing up, one-fourth of 3684 is 921.

Let us attempt a more difficult problem with larger divisor in this way:—

```
40)64960(1624
   40
   ---
   249
   240
   ----
     96
     80
     ---
     160
     160
     ---
```

Explain in the same manner as above, one-fortieth of 64 thousands is 1 thousand, with 24 thousands remainder. 24 thousands equal 240 hundreds. 240 hundreds plus 9 hundreds equal 249 hundreds. One-

fortieth of 249 hundreds is 6 hundreds and 9 hundreds remainder. 9 hundreds equal 90 tens. 90 tens plus 6 tens equal 96 tens. One-fortieth of 96 tens is 2 tens and 16 tens remainder. 16 tens equal 160 units. One-fortieth of 160 units is 4 units. Therefore, one-fortieth of 64,960 is 1624.

A few simple problems of this sort may now be worked out; as, $860 \div 20$, $9600 \div 60$, $7850 \div 50$, $7500 \div 500$.

These simple problems give the process in its simple form and in closest possible relation to the familiar work of short division.

We may now attempt something a little more difficult.

```
12)7740(645        18)7830(435
   72                 72
   --                 --
    54                 63
    48                 54
    --                 --
     60                 90
     60                 90
     --                 --
```

The main difficulty is in learning to use such numbers as 12 and 18 as divisors.

More difficult problems may then follow, as dividing by 25, 36, 84, 230, 460, etc. As the difficulty

grows greater with the increasing size of the divisors, it is necessary to work a great many well-graded problems before the process grows familiar. Fall back upon actual measurements where necessary. It is evident that the understanding of the process must be based upon the analysis (as above) of the very simple problems with which the process is introduced. The explanation of larger problems is then not difficult.

In dividing by numbers as large as 6489 or 7684, it is necessary to use the number expressed by the first, or first and second, digits of the divisor as a trial divisor as measured into the first part of the dividend.

A comparison of the method familiar to the children in short division with the process employed with numbers as they grow larger will show a strong resemblance. As the divisors grow larger, it is necessary to write down those products and remainders which in short division are carried in the mind. With large divisors, it is necessary also to estimate approximate results by means of trial divisors, but underneath all this there is the same process of division, identical in all problems.

It is not now customary to have children learn

a rule for long division, and yet it is desirable that they should see clearly and express in their own way the chief steps in the process. They should see that in all problems, whether in short or long division, it is necessary to follow out a series of partial divisions. Each of these partial divisions involves a divisor, dividend, quotient, product, minuend, and subtrahend. Where the process is complete, the partial quotients constitute the complete quotient.

A formal definition to be memorized by the children does not seem necessary. On the other hand, a mere mechanical routine is not satisfactory. The medium requirement, which is altogether desirable, is a close insight into the meaning of every step in the division, and a discovery of the identity of the process in all forms of division.

It is quite evident that a child will never discover this identity of rational steps in the process except by working out a number of problems, beginning with very simple ones and gradually increasing in difficulty and complexity. This process will extend through a number of lessons, and will involve distinct comparison of examples in short and long division, and a developing clearness as to the fundamental nature of the process.

The chief purpose in memorizing a carefully formulated rule (where this is done) is to clarify the process, to make it more simple and transparent. But such a rule is so technically verbal and abstract that a child is in danger of becoming confused, so that he takes refuge in a verbal memorizing, with but a faint perception of the meaning of the words. Such work requires much time and worry, and does not greatly help in rationalizing the process.

We shall do better to be satisfied with such an explanation of the process as shows that a child has clearly grasped the meaning of the steps in division. This can be done by brief questions and answers. Several of our best recent arithmetics give no rule for long division, but leave it to the teacher to test the insight of the children by less exhaustive and exasperating methods.

A suggestive warning should be given here against an early introduction of the rule after but a few problems have been worked. It was formerly customary to introduce the rule early, as a means of defining the process to be used in working problems. But this led to a blind figuring by rule, which did not satisfy any true purpose.

It seems very desirable in normal schools, where

the logic of processes is carefully formulated by adult minds, that this sharp verbal analysis and nice discrimination in speech should be practised; but to impose this extremely difficult thing upon young children first grappling with these processes is an illustration of how method can run to seed in bad ways.

Our discussion of long division has led us thus far through the inductive movement to such a formulation of the process of long division as we may think desirable. The arithmetics usually give a series of applied problems, taken from practical life (measurements, business, etc.), by which the full mastery of this process can be tested, *e.g.*, How many acres of land, at $86 per acre, can be bought for $9658? Such problems grow more complicated, and are mixed in with other elements (fractions, decimals, etc.), which add to the complexity. These problems give us the final practical application of our process wherever it enters in as one of many elements to be considered in practical affairs (deduction).

In all the later work of arithmetic long division constantly reappears, so that it is one of those primary processes that need to become easy and almost automatic.

We have worked out this process of long division as an illustration of the inductive-deductive treatment of processes in arithmetic. Summarizing, we may see in this process the following steps: —

1. A distinct aim, appealing to the children.

2. A review of the kindred familiar topic of short division as an introduction to the main difficulty.

3. A presentation of simple examples, worked out and analyzed, these increasing gradually in difficulty till the knotty points in the process are cleared up. On account of the increasing difficulty of handling large numbers as divisors, many lessons are needed in order to master the more difficult problems.

4. A close examination of the steps in the process of division and comparison of short division with long division, then of the simple and difficult problems of long division with one another, will bring out the identity of the process in all.

5. The children should be able eventually to give some brief statement of the central idea of long division as a series of distinct partial divisions, and common to all problems.

6. The application of long division to more or less complicated situations in business life, in geography, in social problems, etc.

The natural development of the processes of arithmetic allows an almost perfect illustration of this inductive-deductive movement. It would be difficult to see how any serious modification of this plan can be followed by teachers without breaking up the natural order and sequence of steps. Nor does this imply any dead level of uniformity in teaching, as in all the details of these steps there is large freedom for individuality and for free choice of means and devices.

Having thus illustrated in outline the main steps in the treatment of processes as large units of thought, we may turn now to a more complete discussion of a few leading features of work which naturally occur in the course of this inductive-deductive movement. Nothing could show better the free play of individuality while still holding to a fundamentally essential movement in instruction.

The leading processes, such as addition or division of fractions, writing or multiplication of decimals, are interesting centres of energetic thought effort. And so in each case it is possible to set up an attractive and thought-focusing aim. This aim, where followed up, involves the arithmetical process. For example, how shall we add three-fourths of a foot and two-

thirds of a foot? To a child just ready to enter upon the addition of fractions this question can be so put as to suggest a very interesting problem for solution, an aim. Likewise the division of a decimal by a decimal may be stated in a concrete problem, *e.g.*, How may a strip of land 46.50 rd. long be divided into single lots each 3.1 rd. wide? Such aims, when shrewdly set up, lead into the heart of fundamental processes.

At each step in working out a process (the whole requiring perhaps several lessons) a strong aim should be set up which guides the work till an answer is found and a new problem or aim springs up. In the process of long division, *e.g.*, we noticed that each new difficulty or new and more difficult phase of the process was set up as an aim. Much depends upon the skill of the teacher in setting up these aims clearly and in a way to arouse curiosity and effort.

After focusing the attention of the class upon some valuable aim to be attained, the teacher naturally asks himself the question whether some process already learned may not serve to introduce and partially explain the new process. Short division, for example, was found to be a ready interpreter of

long division, or the multiplication of decimal fractions is found to be easily explained by a reference to the multiplication of common fractions.

This point of contact between the old (in previous lessons) and the new (in the lesson just opening up) is possibly the most vital one in teaching. The teacher who can skilfully manage this interplay between related elements of previous knowledge and the new process to be mastered may be called an expert in one of the chief difficulties of instruction. We will halt at this point to take a look at this crucial difficulty from several points of view.

Young teachers are apt to make the mistake of teaching each subject or process in arithmetic as an independent, self-existent object of study. Perhaps in the minds of some teachers ratio, compound numbers, and square root are as far apart and as unrelated as Julius Cæsar, New York, and Greenland. Wherever this wide separation between the various processes in arithmetic is assumed, right teaching is nullified from start to finish. For arithmetic is the study above all others in which one spirit vitalizes the whole, in which all parts are phases of one thought. Our text-books have a curious inability to reveal this connectedness and

interdependence between all the parts of arithmetic. Many a teacher can teach right through a book and not discover it. This means of course that each process is a distinct routine to be memorized, drilled upon, and applied till it becomes habit. This may explain why so much of our arithmetic work tends to become so mechanical, so thoughtless, so non-rational. Long division, for example, may be looked upon as an isolated routine. Yet an example in long division may involve any product in the multiplication table, any possible sum or difference of elementary addition and subtraction, and any quotient obtained by simple division. In short, there is no fact or number relation in any of the four fundamental processes that may not appear in a problem in long division. This means that at the place where long division is usually taught, in third or fourth grade, a child needs to have at immediate control every fact and process that he has before learned in arithmetic. Beyond this there is almost nothing that is positively new in long division. It consists rather of a little more complex combination of the processes and facts previously learned. Good teachers and makers of arithmetics have not failed to call our attention to this close

continuity and rigid coherency of arithmetical topics throughout the grades. A few statements follow: "Failing to grasp the few simple, underlying principles, the whole subject lacks organic unity in the pupil's mind; at the successive stages of his work he fancies that he is dealing with something new, when in reality he has simply encountered some new phase of that with which he has been long familiar." (Edwin C. Hewett, "Manual of Arithmetic.")

"The lack of recognition of this principle (of unity) largely explains the distaste for arithmetic which may be found in so many schools. Each rule is practically regarded as a bit of subject-matter which somehow has to be got into the child's mind independently of his previous experiences in arithmetical ideas and processes. When, *e.g.*, teachers and text-book writers look upon fractions as having no connection with 'Whole Numbers,' as, indeed, not even to be classed as numbers, is it any wonder that the child, when he comes to fractions, is utterly bewildered, separated as he is from his former number experiences by a break which he cannot pass? For the child learns with what he has learned. When any new matter is presented to him, there must be a breaking up into ideas and

images of all that part of what 'he has learned' which is felt to have a bearing upon the new matter. The selection and adjustment of these ideas is the instrument by which the new is learned; that is, by which it is interpreted and assimilated. The result, in fact, is a remaking of the old — the enlarging, defining, and enriching of the old experiences by means of the new. It is plain, then, that if there is nothing in the old experience connected with the new matter, or if the old experience is only very vaguely connected with it, learning in the true sense of the word cannot take place." (McLellan and Ames, "Public School Arithmetic for Grammar Grades.")

For example, when we begin to deal with factoring, we find that it reaches back into simple addition and subtraction, that multiplication and division are modes of manipulating factors; with a knowledge of factoring one enters the field of fractions fully equipped, while ratio, the decimal system, aliquot parts, involution and evolution, the metric system, and percentage are associations of factors.

John W. Cook says: "If pupils are to become expert in arithmetical operations, they must learn to factor numbers with celerity. Especially is this

the case in Least Common Multiple, Greatest Common Divisor, Fractions, Percentage and its applications, and Proportion. A pupil who has had thorough drill in factoring will perform nineteen-twentieths of the problems in fractions, found in the average written arithmetic, without the aid of his pencil, and few exercises do so much toward giving power to a class as drill of that kind." (John W. Cook, "Methods in Written Arithmetic.")

Upon this idea of the simple continuity and recurrence of the same ideas, Frank Hall based his elementary arithmetic. In his preface he says: —

"The first five lines of this book (page 9) present problems in addition, subtraction, multiplication, division noting the number of groups, and division noting the number in each group. Then, by a kind of spiral advancement, the pupil moves around this circle and upward through all the intricacies of combination, separation, and comparison of numbers."

This vital connection between the earlier fundamental and the later derivative parts of arithmetic suggests that both teachers and children must be kept wide awake in every lesson to much that precedes; that by systematic reviews and drills, and especially by direct association of every new lesson

with appropriate processes of earlier lessons, this extremely helpful and important relation of old and new be kept up.

It is by basing every new lesson upon things which children already know, or, if they have forgotten, must first recall and make use of, that children can do any real thinking, can acquire the habit of working things out for themselves. But in the text-book these vital relations are unseen and invisible; the teacher must read them into the book between the lines.

Merely to go through a text-book without picking up the strings and tying things together is to fail in the most essential part. Later, in discussing the ways of encouraging self-activity in children, we shall find that too much and ill-advised help from the teacher weakens the child in arithmetic. But how is he to help himself if he is not allowed to get his feet firmly planted in a knowledge of earlier processes as a means of interpreting similar but more complex processes later?

At this point we may also refer to the value of oral problems as an excellent means of keeping these earlier facts and processes of arithmetic fresh in mind. Many of the newer processes can also be

best exemplified at first by simple oral examples. The great value of oral work both for review drill and for the illustration of new processes can scarcely be overemphasized.

It is strongly urged by the best teachers that there should be a few minutes (perhaps five) spent each day in a brisk oral review and testing of the simple processes of elementary work, including first the four fundamental operations, and later, work in fractions, compound numbers, factoring, aliquot parts, percentage, writing decimals, etc. This will keep them fresh in mind and ready for constant use.

But it is the peculiar function of the teacher in approaching each new process with the children to revive kindred steps in earlier processes. For the children themselves will not do this. They are not thoughtful in thus using what they have known, but are inclined to look upon each new topic as something totally new and strange. The text-books also fail to give much help to the teacher on this point, and he must constantly supplement the book by hunting out, reviewing, and emphasizing these connections.

This gathering up of familiar ideas in earlier work and focusing them upon the new lesson is some-

times called the step of *preparation*. It follows immediately after the aim of the lesson has been clearly set up. Frequently it can be best done by means of pointed questions which bring old topics clearly to mind.

In the assignment of a lesson upon a new topic, a good share of the recitation period may be well spent in thus bringing up the reserves of knowledge as a preparation for attacking the new lesson. It is in the working out of a new problem in assigning the lesson that an interesting aim for the new lesson is set up and the close connection between the old and the new brought to light. If this is not done, the children will probably spend more time in fretting and worrying over the mode of beginning the work than in the actual process, and without this interpretation of the new lesson through previous ideas they will take on the new process mechanically and without insight into its meaning. In both cases there is great loss of time and effort, and the children are not gaining self-reliant power. Without such introduction, the less capable and over-sensitive children are apt to break down and become discouraged.

Having thus paved the way for the best approach to the new process (*e.g.* division of fractions or meas

uring of areas), we are face to face with the main difficulty, the presentation of the new process. If possible, it is best to attack the new process in a simple typical form rather than in some complex and intricate manifestation. An oral problem, being far simpler, is better than a written one. In introducing percentage, for example, it is better at first to get one per cent of $300.00 than to get $125\frac{3}{4}\%$ of the cost of 648.5 acres at $125.36 per acre. Even after working out such a complex problem, a child does not easily understand the process. Later, when the new process has become familiar, a difficult problem can be easily grasped. Many books, in introducing a new process, give first a series of simple oral problems which can be understood at a glance, and the fundamental idea is thus clearly exemplified. Then follow simple written problems, gradually passing over into more difficult. In this way the attention can be centred strongly at first upon the new process, with the least possible distraction by unnecessary factors.

The examples which are used to introduce a new process should be not only simple, but also, in most cases, concrete rather than abstract. It is better to say "50% of my present cash in money" ($5.00) than "50% of 330."

Most topics in arithmetic when first presented in any and all grades need to be objectively illustrated. As number grows fundamentally out of measurement, these objects should be actively measured with the standard units (yardstick, pound, dollar, etc.).

In intermediate grades there is wide and varied opportunity for a prolific use of the standard measuring units and the various instruments of precision in estimating quantities. The children at this age are just of the active temperament to employ these standards and to be constantly testing and measuring objective realities. The idea is not that the children should squander their time in promiscuously and aimlessly measuring things; but on the basis of definite measurements they shall more clearly grasp arithmetical processes and form the habit of constantly projecting them into the material world surrounding them. This means the steady conquest of the world by quantitive reckoning.

At this point we are most concerned about the mastery of new processes and the illuminating power of concrete measurements as throwing light upon them. In intermediate grades there are a number of these new processes to be worked out in fractions, common and decimal, factoring, reduction of com-

pound numbers, and simple percentage. Actual measurements with standard units may take place, for example, in dealing with groceries, wood, cloth, lumber, freight, coal, brick, fruit, land, liquids, time, distances, speed of travel, cattle and farm products, city blocks, houses and rooms, merchandise. For certain specific purposes paper folding (in fractions) and cardboard construction of geometrical forms (using circle, square, and dividers) are of special value. In manual constructions careful measurement with instruments of precision will furnish numerous illustrative problems, *e.g.* the dimensions and capacity of a box, the area of a room in square yards for a carpet. The barometer, thermometer, and calendar furnish a basis of measuring things otherwise somewhat intangible and yet needing frequent measurement in common affairs.

The arithmetics usually furnish numerous so-called concrete problems, but it remains for the teacher to see whether the children put any concrete sense into such problems. He must see to it that actual measurements with measuring standards at hand in the school be applied frequently to surrounding objects. The application of work to private accounts and bookkeeping gives also a sense of reality. In

short, numerous opportunities are offered for the lucid illustration of new processes by concrete measurements. It is nowhere expected that the text-book will fulfil this requirement. In reality it can do nothing at all in this respect further than give a suggestion.

In introducing a class to a new process, the working out of one or two illustrative problems on the board, by the teacher, is often made a very helpful exercise. It is the teacher's special opportunity for guiding the children to a thoughtful appropriation of the new process. The oral problems preceding have centred the attention upon the chief point (for example, the necessity for changing fractions to a common fractional unit before adding or subtracting). The written problem at the board gives a test of this process with a more difficult problem (as the combined length of the two blackboards, one $7\frac{3}{4}$ yards long and one $5\frac{5}{8}$ yards long).

To keep the class attention so that the wandering thought of even careless children will be held long enough to catch the chief point requires a quick, alert teacher.

This kind of board illustration is especially valuable in assigning a new lesson for seat work. It gives

the children just that introduction to a process which enables them to help themselves in working out a series of problems. Otherwise time and temper are wasted in the friction and discouragement of not knowing what to do, how to get at the problems.

There is, however, a temptation to help children too much, to work out too many problems, to respond to a child's every importunity for help; and some of them are willing to be helped all the time. Some children also need much more assistance than others. But self-help, after all, is the main thing, and helping children too much (a very common fault) weakens and destroys this. In this respect "there is a time to speak and a time to keep silence." The time to speak is when the teacher is working with the class at a common problem which reveals a process. The utmost skill in concrete presentation, in apt questioning, and in holding class attention, is here needed. But when children go to their seats they should be allowed, for the most part, to struggle with their own problems, should be thrown back upon their own resources, even though it causes some friction, trouble, and even failure. The processes of learning have some stubborn elements in them and therein lies their value. The widely varying capacity of

children in the same class makes the teacher's problem far more difficult, but this point we will leave for later treatment.

From the standpoint of the great principle of self-activity we may sum up the matter as follows: Teachers frequently help children too much, and at the wrong place in the process of learning. In a new subject set up interesting aims and discuss the main points with the children. Work out one or two illustrative problems. In this presentation, work inductively, letting children, on the basis of previous knowledge, discover the main idea. In the problems that follow let the children work with as little help as possible. If necessary, only hint at the idea by a question. It is difficult to strike the golden mean between helping too little and helping too much.

When pupils have worked out a few problems independently and begin to feel familiar with the process, it is profitable to ask from them a brief explanation of the steps of the process. Leading up to this, they may write out in brief equations the steps taken; *e.g.* The village has six blocks of paved streets costing, in all, $36,480. At the same rate what will it cost to pave seven additional blocks?

$36,480 = cost of six blocks.

$\frac{1}{6}$ of $36,480 = $6080, the cost of one block.

$7 \times$ $6080 = $42,560, cost of seven blocks.

This does not imply an abstract rule describing the process verbally, but a short explanation revealing a clear grasp of the steps taken in a particular problem. To work out a general rule at this early stage and to express it in exactly formulated language is wholly premature. Some good teachers require no rules at all in intermediate grades, allowing the children to appropriate the processes gradually with repeated exercises. At any rate we will agree that children should become familiar with a process by many oral and written problems, extending in many cases through numerous lessons, before they are called upon to formulate a general comprehensive rule. Children should be allowed to develop gradually toward a clear conception of processes as expressed in abstract rules.

In the matter of explanations by the pupil, David Eugene Smith says: "The period of explanation comes later in the course, say after the fifth grade; but even here the explanation should rather be by questioning on the part of the teacher than by a

full and free demonstration by the pupil. Where complete 'explanations' are required from the pupil, say of subjects like greatest common divisor, the division of fractions, cube root, etc., the result is usually a lot of memoriter work of no more value than the repetition of a string of rules. But by questioning as to the 'why' of the various steps, the reasoning (which in most such work is all that is essential) is laid bare.

"It is the same with many applied problems. The set forms of analysis sometimes required of pupils are of very questionable value. On the other hand, a statement of the pupil's own reasoning is, of course, extremely important, when he is sufficiently advanced to give it. But for primary children any elaborate explanation is impossible." ("The Teaching of Elementary Mathematics," p. 141.)

In fourth and fifth grades we should make clear insight into the meaning of problems and solutions the main thing. Do not belabor and tease children with long verbal or written analyses. They are confusing and discouraging. At every important point question children to discover their insight and sharp appreciation of meanings. The power of verbal analysis grows slowly and may, by premature forcing, cover up ignorance and pretence.

When children work out problems at the board and give short explanations of processes, secure class attention from all pupils to this work. Otherwise there is great waste of time for the majority of the class. It is necessary, however, to place children so that they can see the board work. When necessary, call upon any inattentive child to take up the explanation. This is really one of the best opportunities for teaching by question, by testing, and by explanation. Mere individual recitation, while the majority of the class is careless and inattentive, is poor and wasteful teaching.

In sending children to the blackboard, the teacher should follow a well-devised plan so as to prevent all scrambling and irregularity. There should be close attention to the teacher's words and requirements.

When children are working at the teacher's dictation at the board or at common problems, they often need close attention to secure prompt and independent work. They are apt to copy from one another, to waste time in erasing careless work, or they take on a helpless and indifferent attitude.

In sending a class to the board, let them move

promptly according to a definite plan. At the board, let them obey orders quickly and together. Before writing, see that they have time to image clearly the thing to be written. Produce a strong effort for accuracy and quickness. Have board and seat work clear, neat, and legible.

Accuracy in language and in the written expressions of mathematical operations should be strictly attended to in intermediate grades. As Professor David Eugene Smith says, "It is the loose manner of writing out solutions, tolerated by many teachers, that gives rise to half the mistakes in reasoning which vitiate pupils' work." The looseness and carelessness of many pupils and teachers in oral explanations and in written work are distressing. In the reading of whole numbers, fractions, and decimals, errors abound, so that one could scarcely write correctly from an ordinary dictation by pupil or teacher. It is seldom that one hears a decimal fraction read distinctly and correctly enough to insure against mistakes in writing it. Professor Smith says: "It is only a few years back that such forms as '2 ft. × 3 ft. = 6 sq. ft.,' '2 × 3 = 6 ft.,' '24 cu. ft. ÷ 8 sq. ft. = 3 ft.,' and the like were not uncom-

mon. Now, however, all careful teachers are insisting that such inaccuracies of statement beget inaccuracy of thought, and hence should not be tolerated in the schoolroom."

The following are a few of the warnings and incorrect forms that are gathered from several sources: —

Figures are not numbers.

Numbers are not written "under each other."

$$\begin{array}{llll} 15) & 10^\circ & 45' & 30'' \\ \hline & \frac{2}{3}\text{ hr.} & 3\text{ min.} & 2\text{ sec.} \end{array}$$

The prime factors of 24 are 2 times two times two times three.

$\$36 \times 160$ acres = the value of the farm.

To multiply by 10, 100, etc., add one or two zeroes to the multiplicand.

Hundreds of thousandths.

$20\text{ ft.} \times 20\text{ ft.} = 400\text{ sq. ft.}$

15° of longitude equal one hour of time.

$\$45.00 = 15\%$.

$\frac{1}{4}$ of $\$16 = \$4 + \$3 = \$7$.

As many times as 5 is contained in 20.

Five times greater than $4.00.

$\sqrt{9}\text{ sq. ft.} = 3\text{ ft.}$

$27\text{ cu. ft.} \div 9\text{ sq. ft.} = 3\text{ ft.}$

These and many other similar inaccuracies are very common and are in direct contradiction to that accuracy of thought and statement which arithmetic is supposed preëminently to cultivate.

In this second important step (the presentation and mastery of processes) which we have been discussing from various viewpoints, the chief essentials are self-activity, clearness, and accuracy.

The third step in the working out of a complete unit is usually spoken of as *comparison*. In dealing with the successive problems that are worked out by teacher and pupils in illustrating a new process in arithmetic, this idea of comparison suggests at once an important improvement over our usual method of dealing with these problems. In the multiplication of decimals, for example, after a few oral problems, as .02 of 50 = ? .5 of 300 = ? .15 of 400 = ? we may pass over to written problems such as $2.1 \times .05 = ?$ $3.5 \times 6.3 = ?$ After working out a few such problems, we may stop to look back for a comparison of the processes and results in the problems just worked out. This is not usually done by teachers. But it is advisable, since it gives the children a chance to see a likeness in the work of successive prob-

lems. In each case we find that the problem is explained, as in common fractions, by taking $\frac{1}{10}$ or $\frac{1}{100}$ of the multiplicand and multiplying this quotient by the number of tenths or hundredths. The children may thus discover for themselves that there are always as many decimal places in the product as in both multiplier and multiplicand combined. Without stopping to ask children to make these comparisons, we train them to work blindly and mechanically, and to accept processes from dictation or without clear insight. There is nothing specially difficult or abstract in such comparisons. They are the first steps toward abstraction and generalization. They give children a chance to stop here and there to survey what they have been doing, and to draw such simple conclusions as the problems solved easily bring to light. It is, however, just this thoughtfulness and rational survey that need to be practised at every step in arithmetic. Teachers themselves often draw these conclusions from a single example and call the attention of the children to them. A wiser plan would be to let them spring up in the child's own mind by encouraging him to compare the work of two or three problems.

It is by throwing the children back upon their own power to think in such simple cases that the impulse to self-activity is aroused and wisely directed.

The opportunity for this kind of brief and productive comparison is given in nearly every lesson in arithmetic. Why should we multiply problems in pursuit of a definite idea unless we desire that idea to come to light? As soon as it does come to light, we should either give more difficult problems, or change over to a new order of problems, so as to demand alertness of mind. We can well afford to work a less number of problems and to get more thinking. Merely to work over one problem after another of the same kind in a mechanical way cultivates mental dulness.

The fourth step in the working out of a complete lesson unit is the definite formulation of the process into a distinct rule, or description of the mode of procedure.

The old plan common to the arithmetics was to give the rule at first or after one or two illustrative problems and then make the rule the basis of future solutions. The modern tendency has led to the other extreme and in intermediate grades has omitted

the rules almost entirely. There is no doubt that the general processes of arithmetic should be clearly grasped, and sooner or later, when the time is ripe, should be brought to a clear verbal statement. But in the pursuit of an inductive method we are disposed to put off definite formulation to a later time than formerly, and to seek the shortest and simplest form possible. For example, in square root we now prefer for practical purposes the formula $x^2 + 2xy + y^2$ to the long verbal rule. Brief and concise statements expressing the main points in the generalization in the simplest way are in demand.

If the comparisons mentioned above are kept up in a thoughtful way, the mind of the pupil will ripen gradually to an easy and natural formulation of the rule. "Truly inductive work in arithmetic requires that the rule be put in the background at first; concrete problems should first occupy the attention, and only after several of these have been solved, and the methods compared, should the rule itself be broached and worded. Even then there is a great danger in making altogether too much of memorizing rules, particularly those of operation. Teachers should, however, distinguish between a rule that is originated by the pupil, and one which is dogmatically given to

him. The former has high value; the latter is dangerous." (Smith and McMurry, "Mathematics in the Elementary School.") When the rule is finally memorized, it should carry a full meaning, it should summarize a large and rich experience, simplifying what one has previously learned and supplying an idea with which to work, a tool with which to operate in future emergencies.

The fifth and final step in the elaboration of a lesson unit is known as *application*. In arithmetic this step has usually been regarded as of great importance. The practical problems following every topic, and the miscellaneous problems which close up each important division of arithmetic are applications of previous rules to the affairs of life. A very large share of the time of pupils has been spent usually in working out such practical applications.

On the one side these applied problems are supposed to give a final test of the mastery of previous rules, and on the other hand they lead directly into business life and into practical affairs. Thus may be accomplished the two chief aims which have been set up as the great function of arithmetic, mastery of processes and ability to apply them in useful affairs.

At the present time there is no tendency to

diminish the importance of practical application in arithmetic, but there are two important changes which are recommended by thoughtful teachers. First, is the removal of a large number of the more difficult and complicated problems, the knotty solutions, the conundrums, and curiosities which heretofore have taken so much time and strength of pupils. They used to be regarded as excellent discipline, but they do not add much to the knowledge of principles, and in our present crowded course of study we have not time for them. They will do for specialists, but children are not expected to be specialists in arithmetic. We can well afford to save this time (and it is very large in amount), and if necessary apply it to a better mastery of the simple and fundamental facts and processes, as multiplication table, decimal fractions, etc. We ought to spend much more time in simple oral drills and reviews and far less on artificial, technical, and often unpractical problems of the more difficult sort.

The second important change now recommended is a serious study and interpretation of certain simple industries, economic problems, geographical and scientific topics, and other social interests from the arithmetical standpoint.

We begin to realize that what have been called practical problems are not very practical, that they do not throw much light upon practical life and its difficulties. They are chiefly avenues for mathematical calculation and drill, and yet they are not very suitable even for that.

With the aid that arithmetic lends we may make a very profitable study of a geographical, historical, or industrial topic. For example, the *Hoosac Tunnel* cost the state of Massachusetts $14,000,000. Why did the state pay this vast sum of money to secure a railroad tunnel through the Taconic Ridge, and what were the difficulties involved in the undertaking? A railroad company originally undertook the work at an estimated cost of about two millions. They spent this and then called upon the state for help. The legislature gave them a loan of $2,000,000, and after this was spent, foreclosed the mortgage and took possession of the work. The tunnel was about five miles in length. By means of machine drills the workmen were able to push forward at the rate of 150 ft. per month on the east side, and 90 ft. on the west. At this rate how long would it take to complete the tunnel? In order to hasten the work a shaft was sunk from the top of

the ridge 1030 ft. deep (27 ft. by 15 ft.) to the middle point of the tunnel. What would this cost at a dollar per cubic yard? Boston wished this tunnel so as to have a cheap freight line from Albany to Boston, and thus compete with New York City for the vast commerce of the Erie Canal and the great Northwest. What is the relative distance from Albany to New York and to Boston? How do freight rates by boat compare with rates by rail? Could Boston draw much of the freight, even with the tunnel, in comparison with New York? What are the present populations of New York and Boston? The tunnel was completed in 1875, after 22 years of labor. Has Boston grown any faster since 1875 than before? Compare statistics according to census reports. Has New York grown as rapidly since 1875 as Boston?

The Mt. Cenis Tunnel between Italy and France is nearly 8 miles in length and cost $15,000,000. Why should the cost be so much less? How much cheaper was it than the Hoosac Tunnel per mile? The subway recently built in New York City cost $34,000,000. Compare the population of Greater New York with that of the state of Massachusetts in 1875 and in 1900, and see which cost more

according to population. Population of Massachusetts, 1900, 2,805,346; New York City, 3,437,202. The Erie Canal cost the state of New York originally $10,000,000. What are its length and importance as compared with the Hoosac Tunnel?

This quantitive study of an important geographical topic and the comparison of it with others upon the same standards of cost, size, and utility, cause arithmetical calculations to illuminate a large group of related topics. In this case arithmetic becomes a real handmaid and servant to geography. It is applied arithmetic in the true sense; that is, arithmetic used with the aim of clarifying topics in geography, science, or history, not geography, etc., drawn upon merely to help out the processes of arithmetic.

The advantages of this kind of applied arithmetic are that the problems are easier and more readily handled by the children, that the value of arithmetic as a means of interpreting the affairs of life and of society is brought out in a far clearer manner, and that other great studies receive from arithmetic that important service without which they lose a large share of their meaning, that is, the clear quantitive interpretation.

Large topics suitable for treatment from the arithmetical standpoint constitute great units of study. The centre of interest in such topics passes over from arithmetical processes into life problems, and arithmetic becomes a strong agent for the mastery of these problems from the quantitive side.

The geography, history, and science lessons in fifth and sixth grades are prolific in choice topics which have this pronounced arithmetical side. We need not go out of our way to find them, as is clearly shown in our course of study offered in the following chapter. Moreover, they are in strong need of this arithmetical treatment, for which there is not time in the regular geography, history, etc. Our revised course of study brings into prominence a number of these great units of study with mathematical leanings, and the proper correlation of studies, now so much emphasized, calls for their arithmetical treatment. Such lessons also require a constant and many-sided review of every phase of arithmetic, but they waste no time upon calculations that do not lead to important conclusions. The answers usually given in arithmetic have little, if any, practical value. The

answers found in these applied topics illuminate broadly many industrial, political, and social problems. In this way we get interest and motive behind all our problems. For the answers we seek are worth knowing and remembering.

In this way we get a new interpretation and a strong confirmation of the doctrine of applied arithmetic. Arithmetic interprets life to us through all our studies and experiences. It is not merely a set of processes for intellectual keenness and discipline. It becomes an important instrument for the interpretation of life's problems, both those of the individual and those of society at large. (See the chapter of Illustrative Lessons for other examples.)

In treating the work of grammar grades we shall have further occasion to discuss this phase of applied arithmetic. This completes the discussion of large lesson units according to the inductive-deductive movement. It is clear that arithmetic in the treatment of its processes furnishes almost perfect illustrations of this thought movement.

In the study and broad survey of method in arithmetic, both historically and from a survey of methods in vogue at the present time among good teachers, nothing is more strikingly noticeable than

the tendency to run to extremes in method. Every important phase or method in arithmetic has had its strong advocates, who have pushed it to an extreme. On the one side, object work and illustrative device have been pushed to the limit; on the other, abstraction and rule memorizing have at times carried things with a full sweep. Some have demanded extreme fulness and accuracy of analysis and language statement; others are no less earnest in reducing analysis and technical language to a minimum or to nothing. Some base their whole faith on oral work; some books have almost no oral problems, but abound in written exercises. Some enthusiasts have prided themselves most on the speed and accuracy of oral and written work; others aim chiefly at more deliberate thought work and self-activity in the solution of problems. Many of the arithmetics are loaded with difficult problems; others have almost none, believing that thoroughness in simple processes is the essential thing. We shall find here, as elsewhere, that truth lies between these extremes. True success is to be gained, not by pushing any one of these essentials to an extreme, but by a proper union and organization of all these things into a consistent plan.

It is far easier to specialize toward one of these extremes, — to ride a hobby, — but it is wiser to find some organizing principle which brings each special mode into its proper relation to a larger whole.

Of course each extremist thinks he has found this organizing principle, and at this point the history of method in different periods and countries is of great value in showing what the extreme views and practices have been while the really organizing principles have gradually come into view.

For ourselves we believe that no better general regulative for the whole process of learning has been found than this inductive-deductive movement, which we have attempted to trace out.

In the previous discussion we have found that all these important phases of work, as oral exercises, concrete objective work, written problems, abstractions and rules, analysis of processes, application of principles to practical life, reviews, drill, assignment of lessons, explaining problems, board and seat work — all these find an important place and function. In short, the inductive-deductive treatment of processes seems to supply a fitting place for each important phase of arithmetical method to perform its function.

In addition to the foregoing treatment of method there are a few special topics of great importance that require discussion.

First is the value and use of text-books. A good text-book in arithmetic is necessary in intermediate grades, both from the standpoint of the teacher and the pupils. It serves as a much-needed outline and guide to both. A good text-book in arithmetic not only gives a well-arranged logical order of leading subjects, but grades the exercises and problems so carefully as to enable many children to work their way through many topics without much help from teachers. For seat work and home study the book is essential and saves needless work in copying problems.

Very few teachers are qualified to give a better arrangement and treatment of topics than is given in a good text-book. Not a little help and suggestion are given in most books on method by illustrative problems, by forms of analysis, by diagrams, and by explanatory notes.

One of the most serious faults of teachers is that they do not give a careful study to the whole plan of the author, so as to discover his point of view and method of treatment. Without this the

book fails of the proper effect which it may be well calculated to produce. It would be a good thing generally if teachers could be definitely instructed by the author in the mode of utilizing any given text-book.

There are certain defects and necessary limitations in text-books which deserve our attention.

Generally speaking, there are not enough oral problems. In nearly every new lesson oral problems should lead the way, illustrate the new process, and furnish the real basis for a sound understanding. If such oral problems are lacking, they should be supplied by the teacher. They should also deal with concrete things familiar to the children. In this respect the thoughtful teacher in any given locality can do better than the book, for its problems are necessarily general for all schools.

Nor can the text-book do much in the way of concrete illustration of processes, such as measuring areas, diagramming figures, folding paper to illustrate fractions, imaging situations, etc. The thoughtful and resourceful teacher must make up for these necessary deficiencies of the text-book. The local surroundings and experiences of the

children may supply much illustrative material which the teacher should know how to make use of.

It is often the case that books are not well graded, and this is a very serious defect. The hard problems are sometimes given first, and the easy ones later in a series. If the teacher does not notice this in the assignment of a lesson, he may completely discourage the children.

One of the best graded books for children is that of Mr. Frank H. Hall, and yet in his preface of suggestions to teachers he makes this very interesting statement, "Formal rules are omitted entirely, and the uniform direction to the teacher is, 'If the child cannot solve the problem presented, do not explain, but give him problems that he can solve, and so lead up to and over the difficulty.'" After doing his utmost to make a well-graded series of problems for children, Mr. Hall explicitly states that "the amount of oral work (preceding the book work) may be, often should be, many times as much as is given in the book."

This suggests that there is very great danger of moving too rapidly through the book. Children may do all the problems in the book and still

have a very poor knowledge of the subject. No maker of an arithmetic, surely, would claim to give enough oral and written problems to secure a thorough assimilation of the lessons. It is only young or thoughtless teachers who allow children to run pell-mell through a book in this manner.

We noticed in the earlier discussion, and it is worth repeating here, that the text-book fails to show the close relation and interconnection between the successive topics. These underlying sequences and relationships it remains for the teacher to ferret out and bring to the attention of children. The reviews by which the teacher calls to mind the kindred topics in earlier work that have a bearing upon decimals, for example, the book does little to encourage. And yet this is one of the vital points in good teaching.

The text-books at the present time are undergoing considerable modification. As already observed, certain obsolete topics are dropping out, such as troy weight, apothecaries' weight, equation of payments, etc.

The more difficult problems, once the delight of schoolmasters and bright boys, are no longer regarded as necessary.

The Austrian method of subtraction, by which we use the addition tables, as in making change for a dollar, is rapidly gaining ground.

The division of decimals by multiplying both parts by a number that will make the divisor a whole number is a decided relief in working with decimals.

The freer use of the equation and of algebraic methods is greatly simplifying parts of arithmetic, *e.g.* let $x = 100\ \%$ of the money.

In compound numbers the long reductions are a thing of the past. It is seldom that we use more than two denominations, and one of these is expressed as a decimal part of the other. This alone has removed a large number of tedious and useless problems. "The recent change in custom with respect to compound numbers is quite marked. But a relatively short time ago it was not uncommon to see the area of a field stated in acres and rods, while now it is in acres and decimal parts of an acre; lengths were stated in rods and feet, but now in feet and decimals; and, in general, compound numbers were far more extensively used a generation ago than now. . . . When one considers the rarity of occasions for the use of such numbers, by

himself or in ordinary business, he will be convinced that the time formerly devoted to the subject might be spent on other portions of arithmetic, or on other subjects." (Smith and McMurry, "Mathematics in the Elementary School," p. 43.)

Even the common fraction has lost much of its ancient glory. "The growth in the use of the decimal fraction has been so great during the past century, that much of the work formerly necessary in common fractions has now become almost obsolete. Text-books have been rather slow in recognizing these changes, usually being followers rather than leaders in any movement of this nature. It therefore becomes necessary for the teacher to orientate himself somewhat before undertaking the work in fractions *per se*. . . . Now that the common fraction is used in only a few denominations, and those very small, the decimal fraction becoming more common in general, and nearly universal in scientific and monetary computations, there is no longer the practical necessity for any extensive treatment of factoring and of divisors and multiples on the part of children." (Smith and McMurry, "Mathematics in the Elementary School.")

It is plain that the text-book is not to be blindly

followed. It should be thoroughly understood in its plan and method; its necessary limitations should be recognized; and the special function of the teacher in supplementing it, in relating the parts of it to one another, and in working out more fully the inductive treatment of its processes, should be recognized and steadily acted upon.

CHAPTER V

METHOD IN ARITHMETIC

Grammar Seventh and Eighth Grades

THE simpler processes of arithmetic having been mastered in the primary and intermediate grades, up to the seventh, the peculiar function of the grammar school is to provide the various applications of these processes to those large fields of experience which are embraced in the school course as a whole.

But in recent years our curriculum has broadened out so much in the direction of commercial and industrial life, geography, natural science, history, and the manual arts, that arithmetic will have a large task in mastering these subjects from the numerical or quantitive side. Arithmetic is not merely a series of processes, a science of number; it is a means of better understanding the world about us in its various material forms and forces. It is a standpoint from which the better to see through and around a great many important topics. Without the illumination from mathematics a great many important facts and bodies

of knowledge in geography, history, natural science, and practical life remain hazy and not clearly intelligible. For example, in geography, we speak of the large amount of water-power that is now being utilized at Niagara. But this has a very cloudy and mythical meaning till we have reckoned out its value in some definite way, as follows: —

The Niagara Falls Power Company, on the American side, 1¼ miles above the Falls, has opened a service canal, 250 ft. wide at its mouth, inclined obliquely to the Niagara River. This canal extends 1700 ft., with an average depth of 12 ft., and has water sufficient to develop 100,000 horse-power. This water drops 150 ft. into a wheel-pit through steel tubes, at the bottom of which are turbine wheels. On each side of the mouth of the canal above are two power-houses, each with a capacity for 50,000 horse-power. At the bottom of the wheel-pit a tunnel 7000 ft. long carries the water to the Niagara Gorge, just at the foot of the first suspension bridge. Even this statement does not bring out very clearly the meaning of this water-power. Now suppose the children have visited at home a local mill where 16 men are employed, and the power which runs all the machinery is supplied

by a 50-horse-power engine. How many mills of this capacity could be supplied with power by the Niagara Power Company with its 100,000 horse-power plant? (2000.) With 16 men employed in each mill, how many thousand men would be employed in such mills? (32,000.) If each workman represented a family of 5 people, how many thousand people would thus be dependent upon such a power? What city have you studied that has such a population? (160,000.)

Under its charter the Niagara Power Company has the right to use 200,000 horse-power. The Niagara Falls, as a whole, is estimated to have at least 6,000,000 horse-power. What part of the whole is used by the Niagara Power Company? What per cent of the whole? If the whole of Niagara Falls should be made serviceable, how many workmen could be employed in mills and factories supplied with this power on the previous basis of reckoning? This power, converted into electrical force, is conducted over aërial circuits to Buffalo, 26 miles, where it is distributed from transformer stations to factories, mills, grain elevators, bakeries, street-car and electric-lighting companies, and scores of other places needing electrical power. The economy of electrical

power over steam-power for the city of Buffalo would give an interesting series of problems if there were time to consider them.

In geography, history, and science, wherever it is possible to secure a basis of reckoning in some familiar object in the home neighborhood, it is easy by mathematical reckonings to greatly clarify important topics.

Why should arithmetic be allowed to fail in performing this service to the great instructive school subjects, while in a large number of important topics the mathematical point of view is the most helpful and significant one?

Any important agricultural, manufacturing, or commercial topic, as, for example, the amount of land brought under irrigating ditches in the arid states of the West, the cost of irrigation, furnishes a group of interesting and not difficult problems very suitable for sixth or seventh grade.

Many of the applied problems usually given in arithmetics do not throw much light upon the subjects which they touch. In fact they are not designed to give information. They furnish merely arithmetical practice and discipline. But a group of problems which throw some important topic — like routes of

ocean traffic, the value of the Suez Canal to the world — into a clear light is bountiful in real information. Smith and McMurry, in "Mathematics in the Elementary School," pp. 58 and 59, give a partial list of such topics as follows: —

"As the opportunity offers the following newer topics are introduced, the problems taken from current literature: —

"1. The question of the agricultural interests of the country, connecting with commissions, banking, taxes, and transportation.

"2. The questions of fishing, lumbering, mining, manufacturing, and trade.

"3. The question of transportation, connecting with the study of corporations, taxes, agriculture, mining, and banking.

"4. The questions of labor and of labor organizations, and the relation of each to the questions above mentioned.

"5. Modern treatment of the question of government revenues and expenditures.

"6. Modern treatment of the problems of banking.

"7. Modern treatment of problems involving corporations, as suggested above under stocks and bonds.

"These topics are alive and are valuable for every citizen. It will be a better day for the schools when they replace the obsolete subjects to which reference has been made.

"It is important to note that the topics named suggest our broader commercial, industrial, and social life as the field for arithmetic in the higher grades. Thereby elementary mathematics is made to stand on the same plane as literature and other studies, for all these now culminate in rich generalizations. We should expect in each grade, therefore, a *crude formulation of rules of business* rather than rules for arithmetical processes. For example, a study of farming would result in a knowledge of many principles, touching quantity, that guide the farmer; such as the proportionate division of his land for certain crops, the customary amount of stock, the variation in prices that may be expected. A study of rents would acquaint the pupil with methods of renting farms and tenements, the per cent of profit that can be expected, the dangers, and the customary losses. A study of insurance would likewise lead to some knowledge of the way in which risks are calculated, what rates are paid, and the provisions a man should make for the support of his family after his death.

A text-book should word these principles with the same care with which rules for processes have heretofore been worded. Thus, mathematics in the grades would cease to be purely theoretical; but, on the contrary, would lead to an understanding of practical affairs."

To study an important topic in geography or history from an arithmetical standpoint means the gathering of a considerable amount of statistical data and the working out of a whole series of problems whose answers are discoveries and revelations. They show up the power of mathematics to interpret the world. They bear testimony at the most vital points to the significance of certain industries, inventions, and public improvements in the affairs of men.

In the main the statistical data must be furnished to teachers and pupils, and the problems must be set for them to work out; but it is not difficult to secure, and our arithmetics could well afford to gather up and organize such appropriate statistical data. The census and other government reports alone would furnish more than enough when properly culled. At present such material is not available for teachers and pupils, although not a little of the most useful is furnished in the appendices of the grammar school

geographies, and even the children could be directed how to make good use of it.

It should be distinctly perceived by teachers that this demand for a well-planned interpretation of important industrial and social problems from the numerical standpoint is not an accidental or freakish suggestion. It is almost a necessity of the general course of study which we have laid out in other studies as well as in arithmetic. In geography, elementary science, and history, without any special regard to arithmetic, we have already selected as foundation matter those few great topics which are organizing centres of study. It turns out that many of them have a pronounced arithmetical side, but there is not time in these studies to interpret or work out this point of view. The burden of this task naturally falls to arithmetic, and it will be a source of great strength to the arithmetic itself.

We have here a first-class illustration of what we have observed in other groupings of studies in the curriculum. Great central topics are illuminated from various sides by different studies, and this focusing of several studies for a time or at different times upon a single large unit of knowledge is in full accordance with the best ideas of organizing

and simplifying the course of study. This problem is fully discussed in the introduction to our course of study in the eight grades.

To make this clear, refer a moment to the case of Niagara Falls. The geography naturally gives the general description of Niagara, its physical, commercial, and manufacturing importance; the natural science lessons deal with Niagara specially from the point of view of its geology, its water-power, turbine wheels, and electrical apparatus; arithmetic, as shown above, works out the practical value and effects of its water-power. Each of these studies throws a distinct and special light upon this large subject, and it would be wholly impracticable for one study to teach all these phases at once. Each study reviews and emphasizes or interprets the work of others; and by combining their forces upon the great central topics of instruction, they can produce a very strong and lasting effect.

Looking at the course of study in the grammar schools as a whole, it seems quite appropriate that the mathematics of this period should deal with the large topics of government, commerce, national production, world geography, and large social economies. The statistical material is not difficult, the

problems need not be abstruse or complicated, as is shown in the cases already submitted.

On the other hand, it may be said that the complete arithmetics, until recently, have given a body of miscellaneous and applied problems which are too difficult for grammar school children. One reason for this is that the grammar schools have inherited an arithmetical régime which was for many years regarded as the backbone of a rigid training for young people between the ages of fifteen and twenty, in academies and country schools. When a boy or girl of sixteen or eighteen years could spend three-fourths of the school time and his leisure at home in working out such hard problems, doubtless an excellent discipline could be secured. But the conditions are now wholly changed. The children are far younger (twelve to fourteen), and less mature for such severe effort, and other important studies take up a far larger share of their time and available energy. It is wholly unreasonable to expect grammar school children to do now what much older children once did under very different conditions.

We may say, therefore, that this class of difficult and complicated problems should be left out of our

arithmetics. "In general all this work should relate to the problems of daily life, rather than to tedious and artificially difficult problems intended merely as tasks." (Smith and McMurry, p. 66.)

In conjunction with this large purpose of giving a stronger and deeper mathematical interpretation to the whole range of grammar school subjects, two other important aims should be satisfactorily worked out.

If we inquire into the sources of weakness in the grammar school arithmetic, we shall be able to sum them up under two heads: (1) carelessness and inaccuracy in fundamental operations, (2) lack of self-reliant power to grapple with difficulties in the solution of problems. Both these weaknesses are clearly marked in grammar schools. Whether the causes lie in the lower grades or in the grammar schools need not concern us for the present, but rather the question how to strengthen the upper grade work in these two points.

First the inaccurate and careless use of numbers needs a strong toning up by vigorous oral work. It is in the oral work that wide-awake attention, accuracy, and speed in processes can be gained. No amount of figuring at the seat will give so much vim

and intentness as class drills in oral work by a skilful teacher. In grammar grades the children can stand the pressure of this sort, and the enthusiasm and healthy rivalry of the children will greatly aid. A few minutes each day in judiciously selected oral work will bring about a marked change. "Mental arithmetic, systematically taught from a rationally prepared text-book, is the life and soul of rational method. There is constant adaptation to the normal mental action of the child. During the lesson the teacher is in vital touch with the child's mind; sees the child's personal self-activity in the making of images and in controlling their movements. There is hence the least possible waste for both teacher and pupil. The teacher takes care of the image, and then the concept takes care of itself. From long and varied experience, both in teaching the subject and inspecting the teaching of others, it is firmly held that, compared with 'written' arithmetic alone, mental arithmetic, if systematically taught, will produce at least twice the knowledge and twice the power in a given time." (McLellan and Ames, "Mental Arithmetic.")

A great variety of devices may be used in oral work. The following, suggested by Dr. John W.

Cook, years ago, in his "Methods in Written Arithmetic," are as good now as then: —

"1. The teacher stands before the class and pronounces eight or ten one-place numbers as rapidly as the class can work 'mentally.' When his voice falls, the pupils write the result. The same exercise is repeated until six or eight problems have been performed, the teacher meanwhile keeping a list of the several sums. The results should then be examined. This exercise need not occupy more than three minutes. Pupils should not be permitted to see each other's work.

"2. Send the class to the board. Give several dictation exercises in two-place, or larger, numbers, the pupils writing all the numbers.

"In order that they may get no aid from each other, let them number in order, and assign one problem to the odd numbers, and another to the even.

"Permit no erasing and no changing. Hold all responsible for the first result. Insist that the work shall be scrupulously neat. Carry the work to the right of the point.

"3. Write upon the board a line of figures, and one figure below the line, thus: —

3 8 9 6 7 4 5 6 9 3

8

"Require one pupil to give the sum of 8 and 3; another of 8 and 8; and so on around the class. This should go very rapidly.

"The figures may be written as follows, and another be placed within, thus:—

3

4 8

6 9

9 6

7 9 7

3 4

2 3

5 8

1

"By this means the exercise can go on indefinitely without interruption. Pupils should not be permitted to name the numbers added, but should give the results only.

"Introduce competitive exercises. These must be used with care, as some pupils are easily confused or discouraged. Multiply exercises until the pupils are stimulated to do considerable practice work out of the class."

In treating oral work in intermediate grades we suggested the close connection of oral lessons with written lessons. The oral work supplies usually the best illustrative and simple matter with which to introduce new processes. From these we pass on easily to written problems. But in grammar grades the chief use of oral work is in reviewing and drilling upon processes already familiar.

Mr. Cook calls attention to the fact that a large share of the problems in fractions and in other subjects which are usually worked with paper and pencil could be worked orally. In fact they are much quicker and better worked orally. Where a child takes five or ten minutes at the board to work a problem which he could easily solve in half a minute mentally, he is not merely wasting time, he is learning to dawdle, to labor mechanically, to depend upon his fingers rather than his brain. In the schools often there is much of this needless written work, *e.g.* What are the prime factors of 360? A usual

method is to set the number down on paper and tediously divide by 2's, 3's, and 5. But at a glance a child can see that $360 = 9 \times 40$; $40 = 8 \times 5$. Therefore, the prime factors by quick inspection are seen to be three 2's, two 3's and a 5 — a half or a quarter minute's work.

The most discouraging thing in grammar grade arithmetic is the flabby helplessness of many full-grown children in attacking difficulties. They seem to have lost confidence in themselves or never to have experienced the feeling of power or resource in the self. Intellectually no worse result could appear. The original thought power, the readiness to grapple with difficulties, the eagerness to wage war with a new problem without help from the teacher, all this betokens a strong, inquisitive spirit. Many good teachers in arithmetic are able to get just this strong and valuable result. To a principal or teacher working in grammar grades, the lack of it is a sore disappointment. Curiously this fatal weakness shows itself strongly in our carefully graded schools, well equipped with apparatus and trained teachers. How to react against this feebleness and to arouse a strong and self-reliant spirit in boys and girls is a vital question for grammar school teachers.

As nearly as we can estimate the causes, this mental lassitude and irresolution are due partly to an instruction in large classes, where a rather low average ability is assumed and the teacher does a great deal of illustrating and explaining. Many naturally capable children get into the habit of being fed on an easy diet. Unconsciously, as Colonel Parker used to say, "they are helped into helplessness." Children need some help in nearly every lesson, but one of the best kinds of help that the teacher can furnish is oftentimes the discovery that they need no help. The main business of the teacher, after all, is to see that the children are made conscious of their own resources. In many cases, therefore, direct help should be made with the greatest caution and reserve, not lavishly, and not the same to all members of the class. The teacher should never forget in this connection that he is struggling with the birth of ideas, and the child's own mind must generate them.

One of the most delicate and diplomatic problems of the teacher in instruction is how to wisely help children. By natural temperament nearly every teacher runs to one extreme or the other, either helping too much or too little. Kindly and good-

natured teachers are always on tap at the least suggestion or inquiry of pupils. Strong and unsympathetic teachers reject all advances, throw back every burden upon the pupils, and let them work out their own salvation. Neither of these extremes shows much skill or practical wisdom. They are crude and indiscriminating. Some teachers take a pride in weeding out their classes, on the clear presumption that part of a class are weeds and briers to be gotten rid of, and the rest, a saving remnant, to be pushed to an extreme of proficiency, which will reflect credit upon the teacher. It is a brutal process of getting rid of those whom one does not care to take the pains to bring to a full development of their moderate or even deficient powers. The abler pupils require, indeed, a wider range among more difficult problems, and, while they are engaged in these severer tasks, those of a less ability need more careful attention, not necessarily more direct help, but more time to think, more kindly treatment and encouragement.

Arithmetic was Pestalozzi's favorite subject and the one in which he achieved the most marked success; but the testimony of the school board at Burgdorf was that the remarkable thing about Pes-

talozzi was his success in developing to a surprising degree the ability of feeble children whom others had despaired of.

Teachers of mathematics sometimes seem to forget that children have an emotional nature, and the more their emotions are unpleasantly agitated the less capable they are of strenuous and exact mathematical labor. There is no particular reason why a teacher of mathematics should put off his humanity upon entering his classroom, narrowing himself down to the small dimensions of a mere machine, when the only way to teach mathematics successfully is to take on the full round nature of a human being in order to deal with human beings from the mathematical point of view.

The standard and oft-repeated criticism of grammar school pupils by the teachers themselves is that they do not stop to think, that they begin to figure on a problem before they have clearly grasped its conditions and meaning. It is interesting to observe here that the difficulty in the case is not mathematical, but logical. The mathematical part is the figuring. The understanding of the conditions of a problem is a perception of things not in themselves mathematical. For example: —

Mr. Lewis spent $124.50 for clothing, $275.40 for groceries, and had $134.16 remaining. How much had he before he purchased the clothing and groceries?

Before any operations are performed it is necessary to think clearly the conditions of the problem and to ask one's self the main question that brings all the parts into close relation. The child thinks to himself, "A man starts with a certain sum of money. He spent two sums and has a certain sum left. To find out what the entire sum was with which he started, one must bring these three amounts together." As yet he has performed no mathematical operations, but he has clearly perceived the conditions of the problem, and has discovered the reason why a certain operation should be performed. This preliminary survey of a whole problem to get the facts in their proper relations and to focus them on the main question is what we call the logic of the situation. After this follow the mathematical operations.

Now it is the testimony of teachers that this preliminary survey, this intelligent grasp of the whole situation, is the chief stumbling-block in working problems, and it is in giving unwise help to children

at this juncture that teachers undermine their self-activity and independence of thought. Or, on the other hand, by refusing the most necessary and tactful help less capable pupils are discouraged.

It is sometimes said that children cannot read a problem intelligently, so they need a lesson in reading, in thought interpretation, before putting pencil to paper. All this is true, and only brings us a little closer to the question — how to get children to do this kind of logical thinking.

We naturally fall back upon clear imaging as a basis. To aid this we use diagrams. We also appeal to personal experiences in buying and selling. Practical, commonplace, local knowledge as illustration is of great service. It may be necessary to fall back upon actual measurements, even in grammar grades, for that may have been neglected earlier. It is a practical use of every-day knowledge which is in demand. Questions which cause children to recall vividly their own observations and experiences, and to substitute them in the given problem, lead to self-help.

To simplify the situations still more we fall back on simple oral problems which involve the same

conditions. A boy spent 5 cents for paper, 2 cents for envelopes, and had 10 cents left. How much had he at first? One of the most useful and common methods of cultivating this power of imaging situations is to ask children to make up such problems. This is now much done in primary and intermediate grades.

The kind of thought required in grasping problems is evidently more difficult than figuring. One reason for this is that every problem is a new combination of elements, requiring a certain degree of originality and constructive power to make it clear. All the processes of figuring, on the contrary, are long since familiar, and the mind runs through them with less strain, automatically.

We can afford, then, to spend more time and work more cautiously with children when they are struggling with the general thought side of a problem. In the routine work with processes we can afford to press for accuracy and speed. In this thought work we can "make haste slowly," deliberate, or rather let pupils deliberate — that is, give their minds a chance to work.

Real thoughtfulness is at a high premium in all studies, and the cultivation of self-reliant power in

solving new problems is, in arithmetic, the particular spot where this more difficult achievement is attainable.

There is room for believing that the more systematic application of arithmetic to the important interests of geography, history, and the large undertakings of public and private life, as previously illustrated in this chapter, will open the way for better thought work, for greater intelligence and self-reliance in dealing with practical affairs. The reason for this belief is that such study of large topics from the arithmetical side reveals clearly the power of arithmetic to interpret the world. Thus the interests and motives for the mathematical investigation of large topics are stimulated, and thus we see that the foundations are laid for a strong interest in the thought work that precedes the arithmetical processes.

This grouping of calculations upon the very topics which are found most profitable in other studies reenforces the arithmetic with a background of still stronger interests. Our present isolated problems, not leading to important conclusions that throw light upon great topics of study, give little motive for the best thought work. And it is this motive, springing

out of valuable studies, that leads best to independent thought effort.

Using our present arithmetic as an outline and guide, it is not feasible to put in execution immediately the full plan of applying arithmetic to the large topics of the whole school course and of practical life. But the transition to this plan can be gradually worked out, and several illustrations are given in this book of the reasonableness of such a plan. See chapter of Illustrative Lessons.

It will not be difficult for teachers to spend more class time in discussing the larger topics, such as the business of insurance companies, the function of national banks, the state and national revenues and expenditures, the statistics of immigration, the cotton crop, the corn crop, the iron and coal production of the United States, and in each case to present important problems to solve on the basis of statistics that may be had. An enlargement of some of the present topics in the books in this direction is not difficult.

In seventh and eighth grades the importance of *reviews*, which cover practically all phases of earlier studies in arithmetic, requires strong emphasis. Incidental to the wide variety of topics in applied

arithmetic, there will be constant opportunity to review fractions, the multiplication table, decimals, factoring, percentage, and the tables of compound numbers. Wherever deficiency appears in these fundamental things, time should be taken to review and drill upon them. It is much more important that these simple elements should be mastered than that much time should be spent upon cube root, the metric system, in elaborate analysis or formal algebra. "However much reviews may fail from stupidity, as is apt to be the case with 'set reviews,' a skilful teacher is always reviewing in connection with the advance work. But there is one season when a review is essential, a brisk running over of the preceding work that the pupil may take his bearings, and this is at the opening of the school year. Such a refreshing of the mind, such a lubricating of the mental machinery, gets one ready for the year's work. Complaints which teachers generally make of poor work in the preceding grade are not infrequently due to the one complaining; the engine is rusty, and it needs oiling before the serious start is made." (Smith, "The Teaching of Elementary Mathematics," pp. 143, 144.)

There are many ways of bringing about reviews

of earlier work. Especially in the assignment of lessons, occasion should be taken to call up helpful tables, processes, and devices in earlier work. In most grammar schools, children will be found rusty on what they have been over, and it is wiser to brush up and scour up the old armor than to plunge forward heedlessly into new conflicts.

Much can be done in seventh and eighth grades to strengthen and deepen the sense of the underlying unity and connectedness of arithmetical processes, which we discussed somewhat fully in connection with intermediate grades. In the constant review and application of processes in the grammar school there is frequent opportunity to compare and survey these processes from a more mature standpoint. The difficulty lies primarily with the teachers, who sometimes fail to see these connections. The miscellaneous problems at the close of each important subject in the arithmetics call for a wide range of review.

The most difficult question is left to the last — how to handle children of widely differing ability and temperament in the same class? We cannot give a clear answer to it, but merely a few suggestions.

We think, in the first place, that it is a mistake to set up the same standard for all. The temptation to do this is probably greater than in other studies because arithmetic is an exact science, permitting no deviations. But children differ so widely in their natural ability to grasp and handle number relations that the same speed, accuracy, and thought power are not attainable by all. Slow and sensitive pupils, if afflicted with a harsh teacher who knows nothing but perfection standards, break down and are discouraged. Give quicker, abler pupils all they can do, and give slower pupils time to think according to their brain power in this subject.

It is necessary sometimes to put the stronger pupils at work on advanced or special lines, and to devote much time and care to slower pupils.

We are disposed to think that heavy assignments for home work in grammar grades are a mistake. It is extremely easy to assign too many and too difficult problems. It is distressing to children and parents at home, and, if the written work is effectively examined and tested, it is overburdensome for the teacher. The help that children sometimes get at home is frequently a hindrance to the pupil and teacher. A smaller amount of work, more thought-

fully done, would better meet the needs of average children.

In the assignment of lessons for seat work, a brief review may be necessary to put the children in possession of the general facts and conditions upon which the problems are based. Even the working out of a similar problem at the board may be needed to insure this preliminary basis for independent work.

On the whole it may be said that consideration for the varying abilities of children should be a marked trait of a teacher of arithmetic, and if the subject has a tendency to make him a formal drill master, he should assert his nature in the opposite direction.

The following is a summary of the leading points in the previous discussion of method:—

Regulatives

Primary Grades

1. Use abundant and varied measurements and concrete illustration in elementary processes.

The children themselves should use the objects freely and work out the constructions (with blocks) and the measurements (with standard units).

Abandon the use of objects as soon as the ideas are clear, so as to reach more rapid and abstract number manipulation.

2. In primary grades the number work should have a free range within the limits of a child's experience.

3. Perfection in all four operations simultaneously is forced and premature.

4. The starting-point in a number lesson should be some interesting object, not yet clearly defined, but requiring measurement and numerical interpretation.

5. The formation of a series by the analysis of a larger unit, by counting, by addition, etc., gives a mental movement by which attention can be secured.

Later drills, bringing the combinations of the series into irregular order, are necessary.

6. In learning the multiplication table, the natural and easy order should be followed, and the different tables closely compared with one another to discover similarities and identities.

7. Oral number work should predominate in primary grades.

8. Written work at the board and at the seats should acquaint the children with the figures and symbols of operation.

Intermediate Grades

9. Instruction in arithmetic has two chief phases: first, the derivation of the general processes; second, the application of these processes to important affairs.

10. The introduction to any process or unit of study should be an interesting aim which focuses thought in the direction of the main idea.

11. Keep up a close connection between a new lesson and preceding related work. The old processes interpret the new. The failure of teachers to discover this close dependence of succeeding topics is one of the weakest points in teaching. It is the special function of the teacher to bring these hidden connections to the attention of thoughtless children, to review and emphasize them.

12. Illustrate principles at first with simple oral problems, based often upon the measurement of concrete objects at hand. Do this till children work with understanding and ease. In making up oral problems bring in constantly the familiar experiences and surroundings of the children.

13. Give sharp review drills daily in oral work to establish complete mastery of elementary combina-

tions and processes. In this kind of work one can aim at accuracy and speed.

14. A written problem worked out at the board by the teacher, before the children, with questions and answers, is one of the best means of introducing a new process or of assigning a lesson upon a series of problems.

15. Careful and complete forms of analysis should be preceded by the working out of many simple oral and written problems till the processes are first clear. The main point is to see that a child is thinking clearly, and this the teacher can find out by definite questioning.

16. In working a class together at the board, have children obey orders quickly. Do not let them write till they have imaged clearly the thing to be written. Produce a strong effort for accuracy and quickness. Require board and seat work to be clear, neat, and legible.

When a child is explaining an example at the board, see to it that the class as a whole is in position to follow it. Work for complete, close class attention.

17. In teaching a new process compare the modes of solving several similar problems (oral and written),

and let the children discover a common movement. This self-activity in thoughtful comparison is much neglected. Instead of it children accept the teacher's explanation and work mechanically.

18. Rules of operation were formerly given too soon. Oftentimes now they are not given at all. But a simple statement after proper comparisons and inductions is desirable.

19. The application of number processes to practical life is one of the chief aims of arithmetic, but over-technical, difficult, and artificial problems should be omitted.

The working out of a group of related problems bearing upon some large unit of study in geography, history, or science gives a much-needed illumination of such topics from the numerical side. Such problems are easy for children and furnish strong motives to effort because of the interesting and valuable answers they lead to.

20. A good text-book is necessary for both teacher and pupils. Teachers should study carefully the preface, plans, and notes of the author so as to discover his general arrangement and controlling ideas.

It is extremely difficult to grade a text-book so

carefully that the children can work the problems without help from the teacher. Additional introductory oral problems, derived from local surroundings, are especially needed as a supplement to the book.

Inexperienced teachers generally move far too rapidly through the book.

The text-book is not to be blindly followed, but supplemented in the oral and inductive work, while it may be modified or reduced in some of its topics.

Grammar Grades

21. The emphasis of grammar school arithmetic should be placed upon the application of familiar processes to the whole range of important topics in the school course which require numerical interpretation.

A group of related problems bearing upon one important topic need not be difficult, and leads to interesting and important results.

22. Carelessness and inaccuracy in fundamental operations are common with grammar school pupils. They may be overcome in part by brisk and varied oral drills. Mental arithmetic is of great value in developing power, accuracy, and quickness.

23. In grammar schools especially children should learn to overcome difficult problems by their own resources. Our graded schools often show a remarkable weakness in this respect.

One of the most delicate and diplomatic undertakings of the teacher is to steer wisely along the middle line between helping too much and helping too little.

We can afford to spend more time and work more cautiously with children when they are struggling with the general thought side of a problem. Figuring is much easier.

The concentration of mathematical processes upon large and important topics, and the working out of interesting and significant results, give strong motive to number work.

24. It is very difficult to avoid extremes in method, to bring all the leading principles to a proper adjustment and balance.

25. Our present text-books, by enlarging some of the leading topics, geographical, financial, economic, etc., should be so used as to help in making the transition to the treatment of larger practical topics.

26. The grammar grades furnish the most varied opportunities for reviewing, mastering, and apply-

ing all the elementary processes, and for a deeper insight into their unity and the underlying coherency of the parts.

27. The teacher should be flexible and rationally sympathetic in adapting his standards to the varying ability and needs of children.

CHAPTER VI

ILLUSTRATIVE LESSONS

The Number 7

1. Call attention to the familiar experiences of children in which the number 7 is involved. Several of the pupils are seven years old. A year ago how old were they? How old will they be a year hence? Two years hence?

How many days are there in the week? Name them. How many school days? Five days and how many days are seven days? How many desks are there in one row in the schoolroom? Seven. How many boys and how many girls in the row? Four pupils and three pupils are how many pupils?

Name any other common things in which the number 7 appears. The rainbow has seven colors. The big dipper in the sky has seven stars. How are they arranged? Make a drawing of them.

Four stars and three stars are how many stars?

2 stars and 2 stars and 3 stars = ?

How many fingers have you on both hands? How many more is this than seven? Five fingers and how many fingers = seven fingers?

In playing marbles, how many marbles do you use? We used to have five placed thus, :·: If each of two players has one to shoot with, how many in all? How many cents must be added to a nickel to make 7 cents?

2. Build up the number 7 with inch cubes and parallelepipeds from 2 to 7 in. in length. Arrange the parallelepipeds in the form of a stair and let the children complete the square, writing the results thus:

6 in. + 1 in. = 7 in.	3 in. + 4 in. = 7 in.
5 in. + 2 in. = 7 in.	2 in. + 5 in. = 7 in.
4 in. + 3 in. = 7 in.	1 in. + 6 in. = 7 in.

Remove the blocks and let the children reproduce the series from memory. Then take down the blocks in succession, getting the subtraction series:

7 in. − 1 in. = 6 in.	7 in. − 4 in. = 3 in.
7 in. − 2 in. = 5 in.	7 in. − 5 in. = 2 in.
7 in. − 3 in. = 4 in.	7 in. − 6 in. = 1 in.

7 in. − 7 in. = 0.

Reproduce this series also from memory. Allow the children to build up and take down the blocks with the resultant series till they experience the facts.

Later, with the 2-in. blocks, let them build up the seven, as 2 in. + 2 in. + 2 in. + 1 in. = 7 in., or 7 in. − 2 in. − 2 in. − 2 in. = 1 in.

Write the number in three forms on the blackboard, thus: ⁚ ⁚ • ⁚, seven, 7. Also,

⁚ ⁚ + ⁚ • = 7.   ⁚ + ⁚ • ⁚ = 7.   • + ⁚ ⁚ ⁚ = 7.

Also thus, □ + △. Four sides + three sides = seven sides.

3. Repeat the series in abstract form both for addition and subtraction: $1 + 6 = 7$, $2 + 5 = 7$, etc.

Drill upon the number combinations in irregular order, as:

$1 + 6 = ?$	$3 + 4 = ?$	$2 + ? = 7.$	$4 + ? = 7.$
$7 - 2 = ?$	$2 + 2 + ? = 7.$	$7 - 3 = ?$	$7 - 3 - 3 = ?$
$7 - 4 = ?$	$2 + 2 + 2 + ? = 7.$		$7 - 5 = ?$

Varied and repeated drill exercises are necessary to master these number facts.

4. Make problems of buying and selling: I have 7¢ and spend 5¢ for an orange; how many cents

are left? I have 7¢ and spend 2¢ for candy and 3¢ for a pencil; how much is left? I am 7 years old, my sister 4 years old. How much older am I than my sister? How old was I 2 years ago? How old shall I be after 2 years more? 7¢ + ? = 10¢.

Let the children make up a few problems of this sort: 7 dimes − 3 dimes = ? 5 dimes + 2 dimes = ? 7 dimes = how many cents? 30¢ + 40¢ = ? 50¢ + 20¢ = ? 70¢ − 30¢ = ? 20¢ + 20¢ + 20¢ = ? 30 + 30 = ? 70¢ − 20¢ − 20¢ = ? 30¢ + 40¢ = ? 70¢ − 20¢ − 30¢ = ?

5. At a later time, when the number facts have become familiar, introduce a lesson on multiplication and division in connection with 7. 2 + 2 + 2 + 1 = 7. How many 2's in 7? 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7. How many 3's in 7?

7 − 2 − 2 − 2 − 1 = 0. 7 − 3 − 3 = 1. 7 − 2 = ? 7 + 3 = ?

Concrete problems should be used to illustrate and apply the facts as in the previous treatment of addition and subtraction.

The other numbers from 3 to 10 can be worked out by a similar process, considering each one as a distinct unit of study, requiring many lessons to work it out.

The Tables of 2's and 4's

What objects have you noticed that may be easily counted in pairs, or two at a time? (1) Suggest eyes and ears and count up the eyes in the class as two eyes and two eyes (of Mary and John) are four eyes, and so on through the class. What other things do you see arranged in pairs for easy counting? Wings of birds, pages in a book. Count the pages in a book by 2's. In four pennies how many 2's of pennies? In six pennies how many 2's of pennies? How far can you count correctly by 2's? Try it. Count up to 20, or 30, or 50 by 2's. (2) Use inch cubes and two-inch blocks in counting by 2's up to 24. Use the abacus likewise in counting by 2's to 24. Let the children use the blocks and the abacus in marking off 2's and counting. Make marks on the board by 2's and count them up, so, II, II, II, II, II, II, II, etc. Count backward by 2's from 24. (Thus far in second grade.)

1. Review on the basis of the preceding.

2. Work out the multiplication table of 2's concretely with abacus or blocks.

Make the series, 2×2 balls $= 4$ balls, 3×2 balls $= 6$ balls, 4×2 balls $= 8$ balls, to 12×2 balls $=$ 24 balls.

3. Work out the times idea with the two-inch blocks and compare.

4. Repeat the series backward and forwards in abstract form. Practice upon the single facts in irregular order till they are fixed in mind. $2 \times 2 = ?$ $4 \times 2 = ?$ $6 \times 2 = ?$ $7 \times 2 = ?$ $3 \times 2 = ?$ $5 \times 2 = ?$ How many '2s in 16? How many 2's in 20? $14 \div 2 = ?$ $8 \times 2 = ?$ $18 \div 2 = ?$

At the board and at their seats let the children work out series of such problems till the combinations by multiplication and division become perfectly familiar.

5. Applied problems. If one pencil costs 2¢, what is the cost of 6 pencils? At 2¢ each what is the cost of 8 apples? of 12 apples? Measure with a 2-ft. rule and get the dimensions of a room. At 2¢ each how many apples can be bought for 12¢? for 18¢?

Let children make up similar problems in division and multiplication. The above is merely an outline of leading steps, and in order to secure mastery will have to be enlarged at every point. A similar plan can be followed with other members of the multiplication table. The 4's naturally follow the 2's. In counting by 4's notice the close similarity and

even identity of parts in the table of 2's and 4's, thus:—

2's	2,	4,	6,	8,	10,	12,	14,	16,	18,	20,	22,	24
4's		4,		8,		12,		16,		20,		24

Later still the eights are found to fall into the same series, thus:—

4's	4,	8,	12,	16,	20,	24,	28,	32
8's		8,		16,		24,		32

The discovery of these repetitions in the different tables proves interesting to children, aids the memory, and considerably reduces the labor of mastering the tables.

The 3's, 6's, and 9's are found to have a similar correspondence throughout. If children discover that $24 = 12 \times 2$, also 6×4, also 3×8, it will pave the way later for a quicker understanding of factoring. This kind of work teaches the children to look for connecting links and identities in number work, and gradually brings to light the underlying simplicity and unity in varied operations.

In counting by fours, use as illustrative materials familiar groups of fours, as table legs, the legs of animals, the sides of squares, the number of quarts

in a gallon, quarter-dollars, four weeks in a month, etc. From these pass over to the blocks and abacus, and then to the multiplication table proper, in abstract form and in irregular order.

The Decimal Scale

For a long time people had no way of writing large numbers so that they could be easily read and understood. The figures which we now use in the decimal scale make it easy for us to read and understand a big number at a glance. The ten digits together with the decimal scale are a remarkable invention or group of inventions for expressing numbers. As the decimal scale is used so extensively, it must be understood.

1. Review counting by ones and tens to 100. How many ones in 10? How many tens in 100? Recall the table of United States Money:—

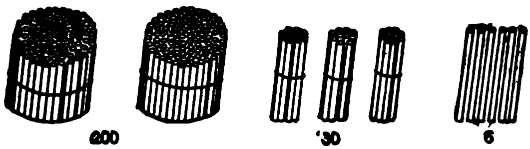
10 cents = 1 dime.

10 dimes = 1 dollar.

Show how we may count out a dollar as 100 pennies or as 10 dimes or as a silver dollar. We are accustomed to say that our United States table of money is based on the decimal scale; why?

2. Our decimal scale of number may be represented by splints or toothpicks.

Make here a picture of single splints, a group of tens in bundles and a group of hundreds in bundles.



These expressed together as one number give 236; six units, three tens, and two hundreds.

By means of the splints and bundles let the children express these numbers, 24, 76, 125, 324, 486, 329.

Let the children count out and bind the bundles of tens and hundreds with rubber bands.

Let the teacher lay out the splints and bundles on the table and the children read and write on the board the corresponding numbers. Extend the scale to thousands, making bundles of thousands from hundreds. Let the teacher lay down the bundles to express 1252, 1452, 964, 1024, 1006; the children read the numbers orally or write them on the board.

Learn the orders and periods from left to right,

2	3	1	4	2	8
hundred thousands.	ten thousands.	thousands.	hundreds.	tens.	units.

3. In English money the scale is different: farthings, pence, shillings, and pounds.

4 farthings = 1 penny, 12 pence = 1 shilling, 20 shillings = 1 pound. Our table of long measure is also irregular,

12 in. = 1 ft., 3 ft. = 1 yd., $5\frac{1}{2}$ yd. = 1 rd.

It is difficult to write and to work problems in these irregular tables. Show this, and compare the decimal with other scales.

The decimal scale or United States money is very easy. \$1.65. Explain what this means and give other examples. This comparison may be too difficult for the first treatment.

4. Repeat the decimal scale to hundred thousands. 10 units = 1 ten. 10 tens = 1 hundred. 10 hundreds = 1 thousand. 10 thousands = 1 ten thousand. 10 ten thousands = 1 hundred thousand.

Learn the orders and periods.

5. Dictate a variety of numbers for children to write, as 604, 390, 600, 5408, 7600, 9230, 10,046.

See that the children think the numbers clearly before writing.

Let the children read numbers correctly from the book or from the blackboard up to hundred thousands. Later develop the writing and reading of numbers to billions.

The Foot-rule and its Relation to the Inch and Yard

What are the common uses of the foot-rule? Name such as you have seen.

1. What is the length of the foot-rule? Make a line on the board a foot long. Is it exactly the right length? How may you tell? What things are commonly measured by the foot-rule? Boards, tables, rooms, garden plots. Children sometimes take their measure with the foot-rule. Illustrate with the children. In this case it is necessary also to use inches. How many inches in a foot?

2. The foot-rule is commonly used by the carpenter in measuring lumber for building. Sometimes he uses a two-foot measure called a square. Sometimes a tape-line, marked off into feet, is used for measuring longer distances, lots, and ropes. The

height of buildings, monuments, trees, church steeples, and mountains is usually estimated in feet. The depth of wells, mines, and water in rivers is also given in feet. Give examples. There are many other common uses of the foot as a standard unit, as in measuring the floors and walls of houses, doors and windows, tables and beds, and the stature of men and animals. Make actual measurements, as of rooms, tables, doors, picture frames, chests, etc.

3. Compare the foot measure with the inch and yard. Why is it necessary at times to use the inch or yard instead of the foot? Give examples of each, as of the inch for measuring the thickness of boards and the size of a brick, and of the yard in measuring cloth, ribbon, etc.

Express inches as fractional parts of a foot. 2 in. = $\frac{1}{6}$ of a foot, 3 in. = $\frac{1}{4}$ of a foot, 4 in. = $\frac{1}{3}$ of a foot, 5 in. = $\frac{5}{12}$ of a foot, 6 in. = $\frac{1}{2}$ of a foot, 8 in. = $\frac{2}{3}$ of a foot, 9 in. = $\frac{3}{4}$ of a foot.

Note also what part of a yard the foot is. Notice the dozen as corresponding to the twelve inches in a foot. $\frac{1}{2}$ dozen = 6. The twelve months in the year also, and the division of the year into quarters, halves, and months.

4. The foot is probably the most common standard used in measuring size, length, distance. In order to measure smaller things the foot is divided into inches, and for convenience in measuring larger objects or distances the three-foot or yard measure, and even the rod and mile are used. In this connection it will be interesting to trace up the history of the yard, as the length of the king's arm, and the careful scientific work in determining the length of the standard yardstick kept in the British Museum.[1] The yardstick and the meter-stick so much used as a standard in Europe should also be compared. Memorize the table of long measure.

[1] "The unit of length is the yard. It was determined in Great Britain about 1760.

"Few pupils have any idea of the care exercised in fixing this standard. Experiments were made at London with pendulums of different lengths until one was found that beat seconds, that is, that vibrated 86,400 times in an average solar day. This pendulum was enclosed in a vacuum to protect it from atmospheric currents, and was suspended at the sea-level in order that it might be as near the earth's centre of gravity as possible in that latitude without going beneath the earth's surface. The length of this pendulum was separated into 391,393 equal parts, and 360,000 of them were taken for a yard. A foot is one-third of this standard, and an inch (from a word meaning a twelfth) is a twelfth of a foot."—Cook's "Methods in Written Arithmetic." For further reference consult the Century Dictionary and the cyclopedias.

5. Test the children as to their ability to estimate the size of rooms in feet, the height of ceilings, the height of buildings, trees, and fences, the size of doors, windows, tables, the frontage of lots, the size of rugs, table tops, picture frames. Test these judgments by actual measurements.

We shall discover later that the linear foot is also the basis of the square foot (with the consequent table of square measure) and this in turn the basis of the cubic foot (with the table of cubic measure), and thus the fundamental and far-reaching importance of the foot as a standard unit comes clearly to light.

Decimal Fractions

In working with common fractions we have found the difficulty of adding and subtracting because of the necessity of changing them to a common denominator, and the multiplication and division of common fractions is also beset with difficulties.

We will inquire whether the use of decimal fractions will relieve us from any of these difficulties.

1. We are already familiar with the use of the decimal scale in working with whole numbers. What is meant by this scale? In treating United

States money we are also acquainted with a table based on the decimal scale. Give the table of United States money. The working of problems in United States money we found much easier than the solution of those in English money, because of this scale. Explain the meaning of this statement.

2. It is not difficult to express fractions in a decimal scale, since it merely requires the use in a new form of a series of common fractions.

In common fractions we are acquainted with $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$. These and other fractions of this series we wish to use in a new form, that is, expressed as decimal fractions. We may observe that there are three ways of writing such a fraction, *e.g.* $\frac{3}{10}$, .3, and three tenths. Likewise, $\frac{13}{100}$ may be written .13, and thirteen hundredths; $\frac{25}{1000}$ as .025, and twenty-five thousandths. The convenience of this mode of writing fractions is in the omission of the denominator, this being expressed by the position of the numerator with regard to the decimal point. Instead of writing $\frac{3}{10} + \frac{4}{100} + \frac{6}{1000}$, we may write .346, three hundred forty-six thousandths. So far as the writing of fractions is concerned, the decimal form is clearly much shorter

and easier; in fact, as easy as writing whole numbers.

In case of need, the splints can be used to illustrate the decimal fraction the same as previously in whole numbers, using the splints singly and in bundles of 10's, 100's, etc.

United States money in cents, dimes, and dollars may be used to illustrate the decimal fraction, *e.g.* \$1.23, \$10.50, \$16.43.

Another advantage of the decimal scale in fractions is that of its similarity to the decimal scale in whole numbers, so that we can express mixed numbers, such as $24\frac{14}{100}$, as one continuous number, 24.14.

Memorize the orders in the decimal scale, and the position of each with regard to the decimal point.

0	0	0	0	0	0
tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths

3. Compare the decimal scale with other fractional series, as $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{64}$, or $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, and it is plain that we cannot easily use such a series with our

present method of writing numbers. When we mix the fractions, as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{7}$, the case is much worse.

4. Our conclusion is that out of all the varieties of common fractions the *decimal fraction*, based on the same scale as integers, is the only one that can be conveniently used by us. It is much simpler both for reading and writing, and for all the operations of addition, etc.

5. To completely master the decimal fraction, it is necessary first to follow out a series of definite and vigorous exercises in reading and writing of decimal fractions.

After thoroughly memorizing the orders and their position with regard to the decimal point, give an exercise in the correct reading of such numbers as 244.65 (two hundred forty-four *and* sixty-five hundredths), 310.25, 300.15, 14.05, 500.05, .32406, 70.002346, 501.2004 (five hundred one *and* two thousand four ten-thousandths). Following such problems, dictate numbers for the children to write at the board. Read very distinctly and correctly, 6001.205, 720.0015, 195.68045. Give the children time to image the number completely before they begin to write. Then require them to write the number continuously from left to right.

The addition and subtraction is so exactly like addition and subtraction of whole numbers that a little exercise is all that is needed.

The multiplication and division of decimals involve distinct difficulties that require a full treatment similar to that in reading and writing decimals as worked out above.

After working out this series of lessons, the full advantages of the decimal fraction over common fractions can be perceived.

Percentage

Percentage is a subject that often makes considerable difficulty for boys and girls. By a closer examination of the meaning and use of percentage, we may find that there is not much that is new or difficult about it.

1. You have heard of merchants selling goods so as to gain 10% or 25% of the cost. Or it may be they have lost 10% or 50% of the cost. What other expressions of this sort have you noticed? Sometimes it is said that 5% or 10% of the people cannot read nor write, or that the attendance at school is but 80% of the enrolment.

There is another way of saying the same thing that you can easily understand. If the grocer should say that he had lost $\frac{1}{10}$ of the value of his goods, you would easily understand. If the boy has two dollars and spends half of it, you know what that means. But 10% is the same as $\frac{1}{10}$ and 50% the same as $\frac{1}{2}$. Give a few examples in finding $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{8}$, $\frac{3}{8}$ of various quantities.

How much is $\frac{1}{100}$ of \$250? of \$600? of 640 acres?

2. Take 100 balls on the abacus (or 100 cubic inch blocks). One ball is what part of the whole number of balls? ($\frac{1}{100}$.)

Two blocks are what part of the whole number of blocks? ($\frac{2}{100}$.) But $\frac{1}{100}$ is the same as 1%, so that we can say one ball is 1% of the whole number of balls, and two blocks is 2% of the whole number of blocks. In the same way, 10 balls are what part of 100 balls? ($\frac{10}{100}$ or $\frac{1}{10}$.) What % is this of the number of balls? (10%.)

In a similar way we may see that 25 balls is $\frac{25}{100}$ or 25% of the whole number of balls, or $\frac{1}{4}$ of the whole number of balls.

3. In order to still better understand percentage, let us notice three different modes we have used for

expressing a fraction. $\frac{5}{100}$ of \$400 = \$20. Again, \$400 × .05 = \$20, 5% of \$400 = \$20.

A man sells $\frac{25}{100}$ of a farm of 200 acres. $\frac{25}{100}$ of 200 acres = 50 acres. Again, 200 acres × .25 = 50 acres. Again, 25% of 200 acres = 50 acres.

4. By an examination of such problems you discover that one idea is expressed in three different ways: (1) as a simple fraction, (2) as a decimal fraction, (3) as a per cent.

It is as if one person had three names by which he is commonly known.

The reason for the use of the decimal fraction, as we have already discovered, is that it is easier to handle either in addition and subtraction, or in multiplication and division.

The reason for using the per cent idea so extensively is that, like the decimal, it is very easy to multiply and divide by 100. It is merely one form of the decimal.

5. The per cent method is applied to a great variety of problems, some of which are suggested as follows: —

(*a*) Oral problems.

There are 500 children in school, 2% of them are sick. How many children are sick? A man raised

300 chickens and sold 7% of them. How many chickens did he sell? 6% of 400 acres = ? 33% of $2000 = ? A man had 500 bu. of apples and 12% of them rotted. How many bushels were lost?

(*b*) Written problems.

A house cost $3500. The architect received 5% of this amount for his plans. How much money did he get?

A farm of 640 acres sold for $55 per acre and the agent received 2% of the selling price for making the sale. What was the agent's fee?

The books furnish many other problems, both mental and written, of this kind.

(*c*) Certain percentages can be best expressed as simple fractions; this gives us the aliquot parts of 100 in percentage, and these are much used in percentage problems; *e.g.* 50% of $40 = $20, or $\frac{1}{2}$ of $40. Likewise 25% of $40 = $10, or $\frac{1}{4}$ of $40.

Make a table of the aliquot parts:—

50% of a quantity = $\frac{1}{2}$ of it.

25% of a quantity = $\frac{1}{4}$ of it.

20% of a quantity = $\frac{1}{5}$ of it.

$33\frac{1}{3}$% of a quantity = $\frac{1}{3}$ of it.

$12\frac{1}{2}$% of a quantity = $\frac{1}{8}$ of it.

$16\frac{2}{3}\%$ of a quantity $= \frac{1}{6}$ of it.

75% of a quantity $= \frac{3}{4}$ of it.

40% of a quantity $= \frac{2}{5}$ of it.

$37\frac{1}{2}\%$ of a quantity $= \frac{3}{8}$ of it.

On the basis of a complete table of the aliquot parts give oral problems daily till they are mastered, as $12\frac{1}{2}\%$ of \$1600 = ? 75% of 16 = ? 40% of 25 = ? $33\frac{1}{3}\%$ of 150 = ?

The above series of steps is a mere outline of the movement in developing a clear notion of percentage and its simplest applications. To give a mastery of the percentage idea a great variety of oral and written problems, partly from the book and partly supplied from familiar surroundings must be worked over.

The other phases of percentage, where a percentage, an amount, or difference is given, or the percentage ratio of one number to another is sought, may be worked out in a similar manner.

In seventh and eighth grades a great variety of percentage problems in geography, history, science, and industrial life will furnish the final application and aid in clarifying other important topics in the chief branches of study.

Population Table

A series of problems based upon a statistical table, given in one of the geographies on the foreign-born population of the United States, follows.

On account of the constant immigration of foreigners to this country from Europe, it is known that we have a very large number of people of foreign birth who now dwell in the United States.

The following table shows the number of these foreign residents from each of the chief countries of Europe. This table may be made the basis of a series of problems upon the populations of Europe and the United States. Dictate to the children this table, or place it on the board for comparative study and as the basis for problems:—

	POPULATION	FOREIGN BORN IN UNITED STATES
Canada and New Foundland	5,032,911	980,938
Germany	36,343,014	2,784,894
England	27,483,400	909,092
Scotland	4,025,647	242,231
Ireland	4,704,750	1,871,509
Norway	2,917,000	322,665
Sweden	5,009,632	474,041
Russia	106,191,795	330,084
Italy	31,667,946	182,580
United States	76,087,350	

Problems

Find the population of the eight countries of Europe which send most people to the United States. What is the number of foreign-born people from Europe now dwelling in the United States? These foreign-born people are what per cent of the population of Europe? What per cent are they of the population of the United States?

The foreign-born Germans in the United States are what per cent of the population of Germany?

The foreign-born Irish in the United States are how many less than the population of Ireland? What part of the population of the United States were born in the United States?

How does the combined population of England, Scotland, and Ireland compare with that of the United States?

The population of Germany is what per cent of the population of the United States?

If there are twice as many children of foreign parents as of the foreign-born people themselves, how many people of foreign birth and foreign parentage in the United States all together?

It is estimated that during the year 1905 one

million immigrants will reach this country from Europe.

If each foreigner from Europe brings with him $24 worth of goods, how much will the foreign immigrants this year increase the wealth of the United States?

How do the foreign-born people of the United States compare in number with the population of New York State? (Consult the population table of states.)

How many cities of the size of Chicago would our foreign-born population make? (Consult in the geography the population table for cities.)

Such problems as the above are suitable in connection with the geography of the United States and Europe, studied in sixth and seventh grades.

Have the map of Europe and of North America before the children while discussing these problems.

These problems give excellent drill in various phases of calculation, and at the same time they throw a good deal of light upon great facts of our national life.

A table of the yearly immigrations to the United States for the last fifty years would be of great interest in a still further series of problems.

Irrigation in the United States

One of the most important topics in the geography of the United States is that of irrigation. It is treated fully as one of the chief type studies in geography. The rapid extension of irrigation since 1880 and the dependence of population upon it, make it a very instructive subject of study for the western half of the United States.

The following statistics bearing upon the subject of irrigated lands, gold-mining, and population in the arid states throw a great deal of light upon the resources of that region and offer a good series of problems whose answers are instructive.

Amount of irrigated land in eleven arid states. (Locate these states in the geography: Arizona, California, Colorado, Idaho, Montana, Nevada, New Mexico, Oregon, Utah, Washington, Wyoming.)

1870	about 20,000 acres
1880	1,000,000 acres
1889	3,631,000 acres
1899	7,539,545 acres

The census report for 1900, Vol. VI, gives the above statistics and also a table of the number of acres under irrigation in each of the eleven arid states. Colorado and California lead all the others

in the amount of irrigated lands and in the products derived from them, as follows: —

	Colorado	California
Total area in square miles . . .	103,925	158,360
Population	539,700	1,485,053
Number of acres under irrigation .	1,299,824	1,158,178
Value of products of irrigation . .	$15,100,690	$32,975,361
Cost of building ditches, etc. . .	$11,758,703	$19,181,610
Gold production in 1899	$19,104,200	$14,618,300

Value of irrigated crops in the eleven arid states in 1899, $86,860,491. Total cost of irrigation system up to 1899, $67,770,942.

Problems

1. What has been the average cost per acre for the system of irrigation in the eleven arid states according to the census of 1899?

2. What is the average value of the product per acre in the year 1899?

3. What per cent of the area of California is irrigated? What per cent of the area of Colorado?

4. Find the total gold production of Colorado and California for 1899 and compare it with the

total product of irrigation of these two states for the same year.

5. The average value per acre of irrigated land in 1899 was $42.53. At this rate what was the total value of the irrigated lands at that time?

6. The average annual cost per acre for irrigation was 38 cents. At this rate what was the annual cost for the whole acreage?

7. At this rate what is the annual expense of irrigating a section of land?

8. What was the percentage of increase in the acreage of irrigation in the ten years from 1889 to 1899?

Family Expense Account for One Month

An ordinary expense account of a family for one month, as given below, may be made the basis of a series of instructive problems.

Such an account may be furnished to the class in mimeograph copies, or the items may be dictated to the class and then used as the basis of problems.

Besides the smaller items the various grocery and meat bills are included in this account. Each of these bills might be made a separate problem. This is also a good place to study the form of

bills, checks in payment, and receipts for payment. A series of such checks returned by the bank, together with the bank account, may easily be used for a separate set of problems.

Family Expense Account for January

Jan. 1	Contribution	$ .60	Jan. 15	Typewriter repairs	$ 12.50
	Fruits and cakes	.90	Jan. 17	Fruit	.62
Jan. 2	Laundry	.58		Hardware	.85
	Ink	.25		Gingham	.50
	Tickets to club	.50	Jan. 18	Butter and jelly	1.45
	Stockings	.75		Service	5.00
	Fruit	.30	Jan. 19	Express	.45
	Pencils	.30		Lawn and thread	.71
Jan. 3	Post Office Box	.40		Lunches	1.00
	Stamps	.63	Jan. 21	Contribution	1.50
Jan. 6	Service	5.00		Listerine	1.00
	Fruit	.60		Malted milk	.50
Jan. 7	Hair cut	.25	Jan. 22	Quinine	.25
Jan. 8	Contribution	.45		Fruits and nuts	.60
	Butter	.75		Castoria	.35
	Tuition	10.00	Jan. 25	Service	5.00
Jan. 9	Livery	2.00		Picture	.40
	Fixing wheel	3.00		Book	1.20
	Post Office Order	1.75	Jan. 28	Contribution	.40
Jan. 10	*St. Nicholas*	3.00	Jan. 30	Dry goods	7.78
Jan. 11	Medicine	.45		Shoes	2.50
Jan. 12	Lumber and paint	1.50		Medicines	.35
	Supper	.75	Jan. 31	Suit of clothes	8.65
Jan. 13	Service	4.00		Birthday presents	2.50
	Envelopes	.53		Meat bill	17.85
	Tennis shoes	.60		Milk bill	5.20
Jan. 14	Contribution	1.00		Eggs and chickens	4.55
Jan. 15	Pair of shoes	3.50		Grocery bill	47.62
	Ribbons	1.20		Rent	22.00

Problems based on this Account

1. What is the total expense for the month?

2. Find what is the average per day.

3. If the family income is $250 per month, what per cent of it is included in this account?

4. The grocery bill is what per cent of the month's expense?

5. The sum of items for service is what per cent of the monthly outlay?

6. If there are seven persons in the family, what is the average expense for each person?

7. At this rate, what is the expense of such a family for a full year (estimating the months as equal)?

8. What is the average expense of each member of the family for a year at this rate?

9. If the family income is $150 per month, how much is the family in debt at the end of a month? At the end of a year?

10. At this average rate of expense, how much does a child cost his parents up to his twenty-first birthday?

City School Report

From the report of the city superintendent, a statement, similar to the following, may be obtained for

almost any city or town. The report of the School Superintendent of San Antonio contains the following items: —

Population of the city (1904)	53,321
No. of children of school age by census	11,841
School enrolment of pupils	8,827
High school	Male 95, Female 231
Expenditures for school purposes	$192,852.01
Teachers' salaries	$107,760.76

Problems

1. What per cent of the population are children of school age?

2. How many children of school age are not in the public school?

3. The school enrolment shows what percentage of the children in school?

4. What is the cost per child to the city for maintaining the schools one year?

5. The salaries of teachers are what part of the whole expense of maintaining the schools?

6. The high school has what per cent of the school children of the city?

7. What is the average cost per each person in the city for supporting the schools one year?

These problems may be extended, by inquiring into the value of the school buildings and grounds, the cost of new buildings, etc.

The system of taxes by which the money is raised for maintaining the schools supplies another series of good problems.

CHAPTER VII

Course of Study

In the following course of study, on the basis of previous discussions, the controlling ideas in the selection and arrangement of topics may be briefly stated thus: —

1. Regular number study is omitted from the first school year, there being an incidental cultivation of number ideas in connection with class and school management and other studies and games.

2. Emphasis is placed upon illustrative devices and measurement with standard units in the introductory treatment of all topics.

3. Oral work is made very prominent throughout the whole course.

4. The course of study is much simplified (*a*) by the omission of obsolete topics and those not needed in modern life; (*b*) by getting rid of over-difficult and complicated problems in all subjects.

5. Constant and thorough reviews are aimed at, and attention is called repeatedly to the inner con-

nection, the underlying continuity, based upon similar ideas and processes, in the leading topics of arithmetic.

6. Clear and correct language and accurate written forms of operation are steadily urged and provided for; but elaborate formal analyses are to be avoided. The necessary simple definitions are illustrated and memorized.

7. Many important topics of geography, science, and history require arithmetical interpretation. The regular quantitive study of these topics is included in the course.

8. The natural overlapping of algebra and geometry upon arithmetic is recognized and made use of only so far as it aids the arithmetical purpose.

9. Thorough mastery of the elementary processes of arithmetic is the fundamental requirement, and the application of these processes to the whole range of knowledge as it gradually comes into view gives the function of arithmetic in the entire school course.

First Grade — Incidental Number Work

In our course of study we have made no provision for regular number work in the first school year. Our presumption is that it is better for children of

this age to gather number experience incidentally from home and school employments. The regular and systematic drill on number combinations in the first year seems to us premature, and the time thus spent can be better employed in widening a child's experiences in nature and in human affairs. With this accumulation of experiences, and with the greater maturity, children may grapple with number more effectively the second year.

The recent widening of the activities of primary children into nature study, school games, literature, drawing, and constructive arts gives a much richer number experience in the first year.

By incidental number work it is meant that where quantitive relations are present, enough attention shall be given to them to make the ideas clear. This is desirable even from the standpoint of nature study, of stories, and of constructive exercises, etc. But this can be easily overdone. It is not our aim to make construction or weather study merely a vehicle for bringing out number relations. The idea is to let number ideas grow naturally, and not to force them.

The following outline indicates a few of the instances where number appears and can receive this incidental attention: —

1. The number of children in the school and in different classes. The relative number of boys and girls. The school enrolment and number in attendance. Absences and tardiness.

2. Distributing and collecting materials for class use, as pencils, books, pens, blotters. A monitor for each row can report the number needed for use in his row.

3. Numbering of children at the board or at the seats. Number of seats in each row or of places at the blackboard.

4. Observe and read the paging of the primer and first reader. Notice number symbols wherever used in any of the studies.

5. In connection with weather study, note clear and cloudy days with colored circles, and work out the record for the week and month. Make a diagram of the thermometer on the board and on paper, and read the markings.

6. In making the clock face, reckon up the hours and minutes. Number of days in the week and month.

7. In the observation of plants and animals, number facts are often of interest, as the number of seeds in pods or parts in flowers, of legs, wings, and other organs in animals.

8. Measuring inches with the foot-rule in constructing seed boxes, play and doll houses, envelopes, and in paper folding, cardboard work, etc.

9. Games which involve counting, tenpins, marbles, dominoes, card games, and any game where a score is kept.

10. Measuring the size of children, calculating ages of children, years and months.

11. Children take pleasure in counting by 1's, 2's, 10's, and 5's, and occasionally attention should be given it.

12. There are many cases where the fractions, halves, thirds, quarters, are used and may require explanation and illustration.

13. Even fables, fairy tales, and myths often bring out number facts.

Professor N. D. Gilbert has worked out more fully this idea under the head of "Related Number Work," upon which the above outline is largely based. See catalogue of Northern Illinois State Normal School, De Kalb, Illinois, pp. 58 and 59.

Second Grade

1. Continuation of the incidental number work of the first year, connected with schoolroom manage-

ment, nature study, manual construction, keeping score in games, counting size and age of children, distance of walks and journeys, garden making, and mathematical games.

2. Complete study of the number space from 1 to 20 by addition and subtraction. Counting by 1's, by 2's, by 10's, and by 5's to 100.

Use common objects about school, home, and neighborhood for counting, as window-panes, chairs, orchard trees, etc.

Notice close connection between 5's and 10's, also between 1's and 10's; *e.g.* each pair of 5's equals one 10.

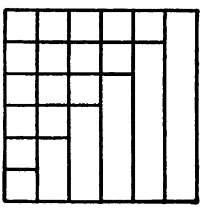
The multiplication series should come later in the year, after the additions have become familiar.

3. Build up the different series from 1 to 10 with inch cubes, and blocks of all lengths from 1 to 10 in. (parallelepipeds 1 in. square at the ends). For example, work out the series based on the number 6, thus: —

$1 + 5 = 6$ $\quad$ $4 + 2 = 6$

$2 + 4 = 6$ $\quad$ $5 + 1 = 6$

$3 + 3 = 6$



Tear down the blocks, also giving the corresponding subtraction series: —

$6 - 1 = 5$	$6 - 4 = 2$
$6 - 2 = 4$	$6 - 5 = 1$
$6 - 3 = 3$	$6 - 6 = 0$

Form similar series with 3, 4, 5, 7, 8, and 9. The individual additions and subtractions can be drilled upon in irregular order after the series have been formed and learned.

The blocks may be used also (the 2-in., the 3-in., etc.), in building by 2's, 3's, 4's, etc., thus leading up to the multiplication table. For the full treatment of the number 7, see chapter of illustrative lessons.

4. In board work by teacher and pupils the number picture, the name, and the figure (symbol) should be written in the same line, as follows, so as to show the "one-to-one correspondence" of these forms of expression, *e.g.* ⁚ · ⁚, five, 5. ⁝⁝⁝, nine, 9.

The number pictures may also be used at first by the children in writing at the board,

as, ⁚ ⁚ + ⁚ · = ⁚ ⁚ ⁚ ·, ⁚ · ⁚ − ⁚ = ⁚ ·, etc.

These exercises give good seat and board work for a short time. Squares, lines, and circles can be used also for number pictures.

With splints or toothpicks let the children lay out the simple geometric forms, as triangles, squares, and rectangles. Also cut out these forms from paper or cardboard and notice the number and relation of sides. Count the faces of cubes and blocks and base simple problems upon the counting of faces, edges, and corners.

5. Make use of the standard units of compound numbers; as pint, quart, gallon, for measuring liquids; foot and yard measurements to twenty feet; dime, cent, and dollar, for measuring values.

Use simple fractions in working with these units; as, $\frac{1}{2}$ (pint and quart), $\frac{1}{4}$ (quarts and gallon), $\frac{1}{3}$ (foot and yard), $\frac{1}{10}$ (cent and dime).

Study of the clock face, and counting by 5's, 10's, 15's, and 30's.

6. Write and use the Arabic figures as they are needed in expressing operations at the board or on paper. Use the signs $+$, $-$, $=$, $\times$, and $\div$ as clear occasion for their use arises.

Learn the Roman numerals to XII so as to read the time on the clock face.

7. As the mind puts number relations into objects in preference to drawing them from objects, the abstract conception of number develops gradually.

In second grade, after the preliminary object work and measurements have laid the basis for clear number ideas, there can be much quick oral work in adding and subtracting of pure numbers, as $3 + 4 = ?$ But whenever the number relations seem blurred, there should be a quick and constant resort to illustrative materials.

8. Every primary school should be well equipped with mathematical apparatus, such as the standard units of liquid and dry measure, quart, pint, and gallon, peck and bushel; also the foot and yardstick, simple scales, clock face, splints or toothpicks, abacus, measured blocks (a full set of 100 cubic inches, ten blocks of each length, 2 in., 3 in., 4 in., 5 in., 6 in., 7 in., 8 in., 9 in., and 10 in.), real or imitation money, good blackboards.

These artificial units are not designed to take the place of other familiar objects, nor is it at all desirable to overload the children with a multiplicity of such materials. But different materials are used for different purposes and there should be a sufficient variety of constructive measurements so as to meet the requirements of early number work.

Additional Explanatory Remarks

1. In both first and second grade children should be allowed full physical activity in measuring with standard units.

Let them also step off distances, play counting games, build with measured blocks, make number pictures at the board or on paper, measure for paper folding and cutting, and write out short statements with figures and symbols.

The abacus or number frame and the splints should be handled by the children.

The purpose of all this is to see that children by the aid of sense perception and motor activity image clearly the objects and groups which suggest number relations.

2. Let correct language be used in describing number operations. Extreme formality in language should give way to brevity and accuracy in describing what is already clearly grasped. Over formality not only gives an unnecessary mental strain but also cultivates a memory of words and phrases that often deceives with an appearance of knowledge.

3. In early number work there should be an emphasis of counting and of addition and subtraction series formed by counting.

The addition and subtraction should precede by some interval the multiplication and division series. Let the notions of multiplication and division grow and ripen gradually. It is easy to carry the memory process beyond the ideas of the children.

Third Grade

1. Complete review and mastery of the number space from 1 to 20, including multiplication and division.

In treating numbers from 1 to 20, form addition and subtraction series as follows: —

7	7	7	7	7	7	7	and other similar
4	5	6	7	8	9	10	additions.
11	12	13	14	15	16	17	

18	17	16	15	14	13	12	and similar
8	8	8	8	8	8	8	subtractions.
10	9	8	7	6	5	4	

Follow these with drills in broken series and mixed combinations.

2. Count to 100 by 2's, 4's, 8's, by 3's, 6's, 9's, and by 7's. Notice the similarity of corresponding series, as 2's, 4's, and 8's; 3's, 6's, and 9's.

2's, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40
4's, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40
8's, 8, 16, 24, 32, 40

3's, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54
6's, 6, 12, 18, 24, 30, 36, 42, 48, 54
9's, 9, 18, 27, 36, 45, 54

See complete treatment of 2's and 4's in the chapter of illustrative lessons.

Follow these series with the multiplication tables in the same easy order, 2's, 5's, 4's, 8's, 3's, 6's, 9's, and 7's. Then break up the multiplication tables and drill in irregular order.

At first and where necessary illustrate with abacus and splints.

3. In the number space between 1 and 100 form such series as the following: —

8	18	28	38	48	58	68	78	88	98
4	4	4	4	4	4	4	4	4	4
4	14	24	34	44	54	64	74	84	94

Form similar addition series.

Drill upon many such addition and subtraction series.

4. Teach and illustrate the decimal scale by the use of splints or toothpicks, forming bundles of 10's, 100's, bound with rubber bands. See chapter of

illustrative lessons for a full treatment of the decimal scale.

Illustrate addition and subtraction of three-place numbers by breaking up these bundles. Also multiplication and division. The abacus or number frame may also assist to explain the decimal scale. Pennies, dimes, and dollars will also help to illustrate the same and give an easy transition to the larger numbers.

5. Train children in reading and writing numbers in units' and thousands' period, as 425,048, and 607,040, etc. Require carefulness in the use of correct language and neatness in board and paper work, making figures large and plain. See that children image the numbers clearly before writing, and memorize the number of each order.

6. *Compound Numbers.*

Review and use the standard units of the second year. Introduce the *pound*, *ounce*, and *ton* (small scales); *quart* and *peck* of dry measure (keep these measures at hand); *minute*, *hour*, *day*, and *month* (clock face and calendar); *square inch*, *square foot*, *acre*, and *square mile*. Measure often with available standard units. Measure and work out the areas of rectangles, rooms, plots of ground, city

lots, gardens. Use the foot-rule marked with inches. Get the fractional parts of the foot, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{3}$, $\frac{3}{4}$, etc.

7. Teach addition and subtraction of two- and three-place numbers. First, illustrate with concrete examples. After preliminary illustrative work, use the following for drill exercises: —

(*a*) Addition	634	375	Subtraction	794	937
	225	423		462	624
(*b*) Addition	537	928	Subtraction	637	829
	264	469		486	764

8. Give daily short oral drills to secure accuracy and speed in addition, subtraction, multiplication, and division. Do not be too quick, but allow slower children time to think.

9. In written exercises for board and seat work be strictly accurate in the use of signs of operation, *e.g.* $6 + 4 \times 3 = 18$, not 30; $8 - 3 \times 2 = 2$, not 10. Errors at this point are very common, both with teachers and pupils.

10. Use the text-book for drill exercises in abstract number work in addition, subtraction, multiplication, and division, at the seat and at the blackboard.

11. The home geography of this year gives great variety of topics for applied number measurements, *e.g. house building*, in measuring basement excavations, rooms, quantities of shingles, boards, brick, stone, sand, nails, paint, etc.

The Garden. — Measuring of spaces, planting of vegetables, yield per acre, price of vegetables, fruit, grain, corn, hay, etc.

The Dairy. — Butter, cheese, milk, cost of keeping horses and cows.

Local Map-making. — Familiar distances measured, making a map to a scale with foot-rule.

Bakery. — Bread and cakes. Cost of flour per barrel, number of loaves per barrel, and price. Measurements in cooking; size and capacity of baker's oven.

Height of hills, towers, steeples, public buildings, and cost of the same.

Transportation. — Drays, wagons and their capacity, wood, coal, lumber, stone, and sand hauled in wagons. Capacity of cars and boats in tons, cattle, hogs, sheep, etc.

Grocery Store. — Selling price of vegetables, canned fruits, sugar, coffee, etc. The family account at the grocery store furnishes a variety of good problems. See chapter of illustrative lessons.

Factories, local shops, and industries offer many excellent practical problems, which throw light upon these occupations.

12. *The Science Lessons.*

Reading of thermometer, the sun-dial; length of day and night, variations in the seasons. Seed production in pods; age and size of annuals, perennials, and trees. Regular number of certain parts and organs in plants and animals, as petals, leaves, legs, feathers, fins, etc.

13. *Number Games.*

Tenpins, marbles, dominoes, card games, arithmetical games, counting-out games. The score at tennis, baseball, golf, and other games.

Distances run or jumped in field-day sports. Races and speed.

Fourth Grade

1. Review of multiplication tables, also addition and subtraction tables by frequent oral drills.

2. Multiplication of two-place numbers by one-place numbers.

(*a*)

$$\begin{array}{r} 43 \\ 2 \\ \hline 86 \end{array} \qquad \begin{array}{r} 23 \\ 3 \\ \hline 69 \end{array} \qquad \begin{array}{r} 22 \\ 4 \\ \hline 88 \end{array}$$

(*b*)

$$\begin{array}{r} 56 \\ 3 \\ \hline 168 \end{array} \qquad \begin{array}{r} 24 \\ 5 \\ \hline 120 \end{array} \qquad \begin{array}{r} 76 \\ 8 \\ \hline 608 \end{array}$$

Multiplication with larger numbers.

$$\begin{array}{r} 25 \\ 32 \\ \hline 50 \\ 75 \\ \hline 800 \end{array} \qquad \begin{array}{r} 67 \\ 32 \\ \hline 134 \\ 201 \\ \hline 2144 \end{array} \qquad \begin{array}{r} 96 \\ 87 \\ \hline 672 \\ 768 \\ \hline 8352 \end{array} \qquad \begin{array}{r} 428 \\ 623 \\ \hline 1284 \\ 856 \\ 2568 \\ \hline 266{,}644 \end{array}$$

Review the decimal scale and illustrate, if necessary, with fuller analytic treatment of the process, *e.g.* $32 \times 25 = 2 \times 25 + 30 \times 25 = 50 + 750 = 800$.

3. *Compound Numbers.*

Table of linear measure (use yardstick and tape-line in frequent measurements).

Table of avoirdupois weight (examine and test larger scales).

Table of liquid measure (barrel, cask, hogshead).

Table of square measure (measuring and reckoning of areas of fields, surfaces, yards, etc.).

Dry measure (measurement of apples, potatoes, etc.). (Now often measured by weight.)

Table of United States money (reading and writing).

A brief historical study of the origin and use of the standard units, as yard, pound, gallon, and dollar is instructive (see cyclopædias).

4. *Short Division.*

(*a*)	3)9	2)14	5)20	6)18
	3	7	4	3
(*b*)	4)19	5)18	9)16	6)32
	4—3	3—3	1—7	5—2
(*c*)	2)64	3)96	4)84	5)1050
	32	32	21	210
(*d*)	5)95	4)76	6)72	8)96
	19	19	12	12
(*e*)	4)824	5)725	8)744	7)686
	206	145	93	98

5. *Fractions.*

Simple fractions expressing the relation of standard units, as inch to foot, ounce to pound, pounds to ton.

By paper folding show that $\frac{1}{2}=\frac{2}{4}$, $\frac{1}{3}=\frac{2}{6}$, $\frac{3}{4}=\frac{6}{8}$, $\frac{1}{6}=\frac{2}{12}$, etc. Show by paper folding that $\frac{2}{3}$ and $\frac{3}{4}$ can be changed to twelfths and added.

Explain the reduction of fractions to lower terms and *vice versa.*

In United States money notice the fractional parts of one dollar, as $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{20}$, $\frac{1}{8}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{6}$.

$16 = \frac{2}{3}$ of what? $750 = \frac{3}{4}$ of what? $96 = \frac{3}{8}$ of what?

6. Reading and writing of numbers to 1,000,000,000. Learn well the periods and orders.

Image clearly the numbers before writing. Give vigorous class drills at the board in writing numbers.

7. *Long Division.*

(*a*) Show the transition from short division to long division, *e.g.*:—

```
   65             16446
5)325           4)65784
  30              4
  --              -
   25             25
                  24
                  --
                   17
                   16
                   --
                    18
                    16
                    --
                     24
                     24
```

(*b*) Long division by 12, 13, 14, 15, etc., to 20.

(*c*) Division by two- and three-place numbers.

By 30, 40, 50, 60, 70, 31, 41, 39, 49; by 36, 73, 86, 94, 324, 860, 940, etc.

(See chapter on "Method in Intermediate Grades" for a fuller treatment of long division.)

Discriminate between measurement and partition, *e.g.* How many yards in 630 ft.? What is $\frac{1}{8}$ of \$250. Show that in measurement the divisor and dividend are of the same denomination, but in partition the divisor is abstract. Work out many illustrations of the same.

8. Provide 100 or more inch cubes. Build up cubes and rectangular solids, and let the children work out mentally the solid contents and the superficial areas.

9. Daily oral work in the review of tables, changing to higher or lower units of compound numbers. Addition and subtraction of simple fractions by changing to a common denominator.

10. Teach the meaning and use of minuend, subtrahend, multiplicand, multiplier, product, dividend, quotient. Make definitions simple.

11. Avoid common errors, such as: —

\$15 ÷ \$5 = \$3.   25 ft. × 3 ft. = 75 sq. ft.

Multiplying the length by the breadth does not give the area. Multiplying together length, breadth, and height does not give the solid contents. Give the correct analysis of these processes: *e.g.* if a floor is 16 ft. long by 14 ft. wide, how many square feet in it.

If the floor were 16 ft. long and 1 ft. wide, it would evidently contain 16 sq. ft., but since it is 16 ft. long and 14 ft. wide, it contains 14 rows of such square feet, or 14×16 sq. ft., which equal 224 sq. ft.

12. Processes applied to other studies and to practical affairs.

(*a*) *Continuation of Home Geography Topics.*

Cost per mile of good country roads; cost of local water-works and running expense; the heating of homes and schoolhouses, expenses for furnaces, coal, wood, janitor or engineer.

The outlay in running a local mill or factory, for machinery, wages, insurance, repairs, raw materials, and losses.

Population of the town; school population; number of voters. Population of city and country compared for the county.

Length of chief railroad lines within the state.

Time and rate of speed for passenger trains. Cost of tickets for certain distances, between cities.

Trip round the world on the home parallel, mark ing and comparing distances.

Capital needed in sinking a coal shaft and for machinery and labor in operating a coal mine.

Other similar topics in geography.

(*b*) *Elementary Science.*

Quantity of weed seeds destroyed by birds.

Distance to the sun and moon and size of each.

Diameter and circumference of the earth.

Power gained by the crowbar, pulley, and rope.

Power of engines, windmills, and water-wheels.

Comparative weight and value of different metals.

Amount of money wasted in alcoholic drinks and tobacco.

The age of plants and trees.

(*c*) *History.*

Length of chief periods in American history. Chief periods of the world's history; time before and since the birth of Christ. Important dates.

Age of cities and buildings; monuments.

Age of the oldest citizens; oldest houses in the town. Centennial and other celebrations.

Fifth Grade

1. *Review of Long Division.*

This involves careful work in all the fundamental operations and shows where oral drills are needed.

2. *Compound Numbers.*

Review previous standard units and tables, with quick drill in oral reductions.

Table of time from seconds to centuries.

Natural units in day, year, and lunar month.

Table of cubic measures; build up small cubes, and concrete and analyze the reckoning of cubical contents.

Cord measure for wood and stone.

3. *Fractions.*

Review of simple fractions by oral work. (See chapter of illustrative lessons in "Method of the Recitation.")

Prime factors of numbers to 25. Prime numbers to 50.

Factoring of numbers to 50 at sight.

Change mixed numbers to fractions, and *vice versa.*

Drill on the oral addition and subtraction of fractions.

Not much written work is needed in fractions.

Multiplication of fractions.

Two ways of multiplying a fraction; multiplying a fraction by a fraction. Division of a fraction by a whole number; division of a fraction by a fraction: (*a*) by reducing to a common denominator; (*b*) by inverting the divisor and multiplying.

Use and define the terms numerator, denominator, factor, prime factor, and common factor.

4. *Decimal Fractions.*

Show that the decimal is another mode of expressing the common fraction, as $\frac{3}{5} = \frac{6}{10} = .6$.

Show the relation of decimals to the decimal scale in whole numbers; show the relation of decimals to United States money.

Read distinctly and write correctly decimals to thousandths. Be careful always in placing the decimal point.

Before writing a decimal, think clearly: (1) the number of orders in the decimal; (2) the number of zeroes, if any, to the right of the decimal point. This will enable the pupil to think or image the number clearly before writing and to write the number promptly from left to right.

Give many dictation exercises in writing decimals. Make clear the rule for marking off the number of

decimal places in the product. (See chapter of illustrative lessons for a fuller treatment of decimals.)

Division of decimals.

The form suggested by David Eugene Smith should be much used. ("The Teaching of Elementary Mathematics," p. 122.)

$2.5\overline{)7.325}$. Change to this:

$$\begin{array}{r} 2.93 \\ 25.\overline{)73.25} \\ \underline{50} \\ 23\,2 \\ \underline{22\,5} \\ 75 \end{array}$$

Give many simple problems in the division and multiplication of decimals to thousandths.

5. *Simple percentage* (see chapter of illustrative lessons for fuller treatment).

Percentage is based upon the simple fraction, hundredths.

(*a*) Get 1%, 2%, 3%, and 4% in numerous concrete problems, *e.g.* 2% of 400 acres. Many oral problems for quick class work.

(*b*) Teach $50\% = \frac{1}{2}$, $25\% = \frac{1}{4}$, $33\frac{1}{3}\% = \frac{1}{3}$, $10\% = \frac{1}{10}$, $16\frac{2}{3}\% = \frac{1}{6}$, $75\% = \frac{3}{4}$, etc. Apply the aliquot parts to the solution of many oral problems.

6. *Business Forms.*

Make out bills as illustrated in the arithmetics. Study actual accounts and bills as made out at the store. Receipted bills, bank checks as receipts.

Private accounts of expenditures.

Secure neatness and accuracy in written papers, but do not have many of them.

7. *Mental Arithmetic.*

Constant oral reviews of fundamental operations. Most problems in common fractions should be performed without pencil or paper.

Give frequent and varied problems in aliquot parts and in simple percentage.

Divide numbers by 10, by 100, by 1000, by shifting the decimal point. Divide also by 25, 50, 40, 60, etc. Most problems in compound numbers should be limited to two denominations and worked orally. Use mental arithmetics and make up mental problems from common experience.

8. Use varied devices for oral and board work; *e.g.*, for addition and multiplication write thus

4, 7, 3, 9, 6, 4, 2, 5, 8

7

Arrange the figures also in the wheel.

9. Avoid common errors in language.

Figures are not numbers.

Figures are not written under each other.

"Will be" and "would be" are not often appropriate forms of expression in arithmetic.

2 tens = 20 units + 5 units = 25 units is wrong.

10. *Applied Problems in Geography.*

The Erie Canal, cost of building, deepening, etc.

Proposed expenditure of $101,000,000 by New York State for enlarging the canal. Expenses, tolls, and freight.

Cost of Hoosac Tunnel and other tunnels.

Expense for subways in New York, Boston, and other cities.

Bridge building: the Brooklyn bridge; number of passengers daily across the bridge; income, expense, and cost.

Capacity of elevators and warehouses, reservoirs, coal-bins, ships, cars.

Expense of the state government: income of the state, salaries of state officers, support of state institutions, state university, and normal schools; the finances of the state penitentiary; support of state militia.

Irrigated lands: extent of lands under irrigation;

agricultural wealth of the irrigated regions; gold production in different states compared, and products compared with that of irrigation; coal, silver, copper, and iron production.

Amount of shipping (tonnage) on the Great Lakes and at chief harbors.

Cotton production, its value, and the amount exported.

11. *Science Problems.*

Extent of injury to apple and other fruits by the codling moth and other insects.

Measuring the height of mountains by thermometer and barometer; air pressure.

Expense of the tobacco habit to the individual and to the nation.

Weather calculations; records of temperature and of barometric readings; evaporation and precipitation; the rain gauge.

Water power at Niagara; electric power.

12. *History Problems.*

Length and duration of Columbus's voyage and other great voyages, as of Magellan, Drake, Da Gama.

Lewis and Clark expedition; distance and time required; expense of the trip.

Size of new continents; length of rivers.

Breadth and size of the oceans.

Chief periods of English history.

Amount of wealth gained by Spain in America.

Sixth Grade

1. *Systematic Review of the Elementary Operations in Arithmetic.*

(*a*) Abundant oral problems in the four processes; multiplication tables.

Fractions: quick mental work in all forms.

Prime factors of numbers to 50.

One-step reduction of compound numbers.

Aliquot parts of 100.

(*b*) Review of written arithmetic.

Short and long division.

Decimals to thousandths in multiplication, division, etc.

Business accounts; notes; bills.

2. *Fractions* (advanced work).

Multiplication and division of fractions.

Factor numbers to 100 at sight.

Reduction of fractions to a common denominator.

Greatest common divisor; least common multiple.

Confine the work to simple problems, mostly oral.

How to find the prime factors of larger numbers by inspection.

Multiplying and dividing a fraction by a whole number.

Explain the inversion of divisor.

3. *Decimal Fractions* (advance).

Read and write decimals to four periods.

Memorize the number of each period and order to the right of the decimal point.

Explain the reason for marking off the number of decimal places in the product and quotient.

Work also by the short method in multiplication and division, as follows: —

$$444.659488 \div 5.3872$$

```
12.36                          82.54
 2.5364                     ──────────
──────              53872.)4446594.88
24.72                     430976
 6.180                    ──────
  .371                     136834
  .074                     107744
  .005                     ──────
──────                      290908
31.350                      269360
                            ──────
                             215488
```

4. *Compound Numbers.*

Cubic measure; table.

Circular measure; table.

Table of English money, and comparison of units with United States money.

5. *Percentage.*

Review of simple percentage and aliquots.

Applications to buying and selling.

Simple interest, with oral and written problems.

Use a mental arithmetic and make up many simple problems based on everyday life.

6. *Analysis of Elementary Processes.*

In this grade children should analyze familiar processes clearly and express them in simple language; *e.g.* addition of fractions, the multiplication of decimals, long division, cancellation, least common multiple, etc. The definition of terms, as partition, prime factor, and percentage, should be understood and memorized.

7. Notation and numeration of whole numbers to six periods. Prompt and accurate reading and writing of numbers from dictation for class work at the board.

8. In this grade, on account of the review of all elementary processes, there is special opportunity

to simplify arithmetic by discovering the similar processes and unifying principles that run through the various processes in arithmetic; *e.g.* the decimal scale is found in whole numbers, in common fractions, in United States money, in decimal fractions, in percentage, and later in all the metric tables. The ratio idea also extends through all numbers.

9. *Applications to Geography.*

A cotton mill: cost of raw cotton; expense for buildings, machinery, and equipment; number and wages of employees; losses from wear and tear, fires, insurance, strikes, competition, and changes in markets; selling and distribution of goods; collections.

Municipal improvements, as water mains, water towers, engines, reservoirs, filtering plant, employees.

Population of ten large cities in the United States compared among themselves and with other great cities of the world.

New York City: chief traffic routes by land and water centring here. Quantity of exports shipped in and out.

A year's expenses of the government for the army and navy.

A great newspaper: cost of collecting news, printing machines and presses; paper used; reporters, compositors, and editors.

A large railroad system, as the Pennsylvania or New York Central lines: extent of lines; value of roads, stations, and equipment; number and wages of employees.

10. *Science Lessons.*

Loss to farmers through chinch bugs and grasshoppers.

Percentage of nutritive material in different foods.

Power generated by hydraulic press, steam engine.

The telescope and microscope and their power.

Amount of fresh air needed in ventilating houses and schoolrooms.

11. *History.*

Size of armies in the Persian and Punic wars.

Size of Indian tribes and populations of the different colonies.

Penn's purchase of Pennsylvania; price and quantity of land.

Number of colonists, as English, French, Dutch, Spaniards.

Extent of territory in the early grants.

Cost of the French and Indian wars.

Seventh Grade

1. *Full Study of Percentage.*

Treat percentage as a case in simple fractions. In using the text-book supply an abundance of simple oral problems from common experience.

Review the aliquot parts and drill thoroughly upon their simple applications.

Require children to explain in all cases per cent of what, and thus avoid confusion.

In applying percentage to any given form of business, be sure to discuss fully the conditions of the business as a basis for understanding the problems.

2. *Commission and Brokerage.*

Study the subject in its present business aspects. Use newspaper quotations as the basis of problems.

3. *Interest, Simple and Compound.*

Many oral problems and simple written problems are better than complex and difficult solutions. Examine and write out the forms of notes and endorsements. Simple problems in partial payments. Show business papers, as mortgages and mortgage notes and coupons.

4. *Banking.*

The business of a bank and its relations to other kinds of business. The vaults and safety

deposit boxes. A personal bank account. Checks and drafts. Savings banks and plans. Interest charged on loans. Interest paid on deposits. The trustworthiness of banks. Bank inspectors. Different kinds of banks. Banking can be well studied in its chief business aspects in almost every community.

5. *Insurance.*

Fire insurance. Rates charged on different kinds of property. Dwelling house insurance. Life insurance. Endowment policies. Annuities. Options. Examinations for insurance. Provision for families. The large insurance companies. Capital. Mutual companies and assessments. Unsafe companies.

6. Instead of the more difficult problems given in many text-books, and especially in the place of obsolete topics, the following subjects are suggested as suited to bring arithmetic into close relation to industry and to important topics in other studies: —

(*a*) *Storekeeping and Accounts.*

Buying and retailing goods, buying and selling prices, per cent of profits, sources of expense in conducting a business, necessary losses, bad accounts, breakage and waste.

(*b*) *Farming.*

Value of farm lands, rents, cost of stocking a farm with animals and machines, cost of barns, granaries, fences, silos, wells, and windmills, bad crops and losses, variable prices for grain and live stock, profit on the whole investment.

(*c*) *Saw-mill or Planing-mill.*

Cost of machinery and mill, supply of logs from the woods and cost, expense for labor and repairs, how economies are practised in using up waste material, sale and shipment of lumber and finishing material, contracts for buildings.

(*d*) There are many other common industries each to be studied from its own peculiar point of view as tested by financial output. Such are a shoe factory, hardware store, canning factory, creamery and dairy, the florist's hothouse, a vegetable garden, coal mining, quarrying, brick kiln, printing a newspaper.

(*e*) Bookkeeping has an important bearing upon all these topics. The method of keeping a private account by a farmer or householder, accounts as kept in a small retail business, are deserving of a few definite lessons.

7. *Arithmetic applied to Geography.*

Finances of the national government. The budget

for one year in its main items. Income through customs and internal revenue. Income from the post-office and from sale of public lands.

Expenses of the government (chief items) for one year, for the executive, legislative, and judicial departments, for pensions, army and navy, public buildings, rivers and harbors, agriculture, mining and fisheries.

Comparison with the expenses of royalty in England and Germany.

The iron industries of England and the United States compared.

Traffic routes of Great Britain with her colonies. Length and importance of these routes. Volume of trade with different countries.

Population of France compared with that of other countries in Europe and America. Rate of increase in different countries.

Expense of keeping up the German army. Compare with the expense account in other European countries. Size of armies in time of peace.

Cost of the common schools in the United States. Number of pupils and teachers. Expense of equipping a large university.

Extent of navigable rivers and canals in Europe.

Number of miles of railroad in Europe and in America. Value of railways and equipment.

8. *The Revolution.*

Number of men enlisted from each colony. Debt incurred by each colony and by Congress during the war. Hamilton's financial plan for paying the expenses of the war. Funding the debts.

9. *Science.*

Forests of the United States. Extent and value. Taxing of alcoholic liquors. Income from such taxes for cities and government. The earth and the planets: relative size, distance, and orbits.

Eighth Grade

1. Review of chief topics in elementary arithmetic. Rapid and accurate work in addition, subtraction, etc., with oral problems.

Review of simple fractions in the four operations. Chiefly oral problems.

Reading and writing and operations with decimals.

Factoring, L. C. M. and G. C. D.

Tables of compound numbers.

Mensuration of rectangles, triangles, parallelograms, circles, etc.

Percentage as worked out in seventh grade.

In the final review in eighth grade, with the greater maturity of the children, there is opportunity to realize more clearly the close connection between all the topics of elementary arithmetic.

A good mental arithmetic may be of much service. Oral problems secure complete mastery of processes and lead to rapidity and accuracy.

2. *Taxes.*

The system of local, county, and state taxes. State laws. The assessor and collector. Purpose of taxes; local, county, and state officials, and the expenses therefor. Public buildings, roads and bridges, schools taxes, public works. Special assessments.

The city budget of New York or Chicago for one year. Bonding cities to raise money for improvements. Cost of city improvements, as parks and streets, water supply, sewer system.

3. *Corporations.*

The organization of stock companies. Certificates of stock, dividends, bonds, and interest. The directors and officers of stock companies. The business of large corporations, as of railroad, mining, and manufacturing companies.

The market for stocks and bonds. Newspaper

quotations and their fluctuations. Broker's commission. Speculation in stocks.

4. The foreign trade of the United States. Exports and imports. Quantity and value of each. Money exchanges with foreign countries. The money units of England, France, and Germany.

The metric system used in Europe. The meter-stick, compare with yardstick. The history of the metric system. Its advantages. Teach the near equivalents of the chief metric units.

5. *Longitude and Time.*

Table of correspondences, 15° corresponding to one hour of time.

Standard time; time belts of the United States. Difference between sun time and standard time. Mathematical geography, size of the earth.

6. *Land Surveying.*

The United States land surveys. Prime meridians, townships, sections, quarter sections. The sale of government lands, preëmptions. The price of agricultural lands, of city lots, of business blocks.

7. *Square Root.*

Involution and evolution.

Use the diagram and the algebraic formula. Extract two and three place roots.

Notice the squares of fractions and decimals.

8. *Mensuration.*

Measurement of the circle. Area of the circle.

The volume of prisms and cylinders.

Capacity of cisterns and tanks.

Pyramids and cones and their volume.

"Pupils should be provided with a rule, a pair of dividers, and a right triangle of wood, hard rubber, or cardboard.

"An accurate diagram, drawn to scale, should be made of all problems that admit of it. The various rules of mensuration should be developed inductively, from actual measurement of objects. They should be expressed in formulas. Simple algebraic processes should be taught as they are needed in the development of formulas." (Illinois Course of Study, page 91.)

9. *Geography Problems.*

The colonial empire of Great Britain. Area and population compared with Russia and the United States.

Comparison of the great rivers and river valleys of the world in area, navigability, traffic, and population.

Asia, area and population compared with other continents.

A line of ocean steamers between New York and Liverpool. Size, cost, capacity, horse-power, and speed of vessels. Expense of coaling and manning a steamship. Cargo and passengers, profits.

The navy of England, its cost. Compare with the United States and other countries.

The great ship canals of the world, cost, benefits, and profits on tonnage. Effects upon the traffic routes and trade of the world.

The Siberian railway, length and cost. Compared with continental lines in the United States.

Population of the world, distribution of races and religions.

The gold and silver, the grain production of the world according to leading countries.

Comparison of the great cities of the world in importance, trade, and population.

10. *History Problems.*

Immigration to the United States from European countries since the Revolution.

The Civil War: expense to the North and South; number of men engaged on both sides; losses.

The growth of the United States in territory.

Growth of the United States in population according to the census reports.

11. *Science.*

The geological resources of the United States in gold, silver, iron, coal, oil, and building stone.

Statistics of the effects of alcoholic drinks upon crime and disease.

Modes of measuring electrical force, meters.

Light and sound and their rate of movement.

CHAPTER VIII

Books for Teachers

The number of books bearing directly upon the teaching of arithmetic is not great. The following are those which the author has found most helpful:

1. "The Psychology of Number," by McLellan and Dewey. D. Appleton & Co. This is the best standard treatise and has a full discussion of the psychological basis of number.

2. "The Teaching of Elementary Mathematics," by David Eugene Smith. The Macmillan Co. This gives an interesting historical survey and discusses also the present aims and course of study. The recent changes in arithmetic and advisable omissions of obsolete topics are indicated.

3. "Mathematics in the Elementary School," by Smith and McMurry, a pamphlet of the series of the *Teachers College Record*, Columbia University. This pamphlet discusses the basis for a course of study and outlines such a course. It views arithmetic

in its relation to the child and to the demands of society.

4. "The Grube Method," by F. Louis Soldan. Chicago Interstate Publishing Co. This is a good introduction to Grube's plan and course of study.

5. "Grube's Outline," by Levi Seely. Kellogg & Co. Is also a good introduction to Grube.

6. "Methods in Written Arithmetic," by John W. Cook. A very useful book for teachers who wish to acquire accuracy and thoroughness in processes.

7. The mental arithmetics are deserving of the special attention of teachers on account of the prominence given to mental arithmetic, both as a means of explaining processes and for varied application and drill.

8. The text-book in use in a school, as already indicated, should be fully mastered in its plan and method by the teacher. But no large number of text-books on this subject is needed.

9. Many of the books of general pedagogy have chapters on arithmetic, as Fitch's "Lectures on Teaching," Collar and Crook's "School Management and Methods in Instruction," and many others.

10. Brooks' "Philosophy of Arithmetic," Sower, Philadelphia, is one of the standard books.

Those wishing a list of German and French books on arithmetic will find what is essential in the last chapter of Smith's "Teaching of Elementary Mathematics."

In order to bring arithmetic into proper contact with business and industrial life, and with important topics in geography, history, and science having a strong arithmetical side, there is need for much more abundant statistical data than the arithmetics contain. These data require to be collected and focused upon important problems. At present these statistics must be sought in the volumes of the census and other government reports, in the statistical almanacs, in encyclopædias, in the appendices of geographies, and in other scattered publications.

It is to be hoped that a book may be soon forthcoming which will give an instructive introduction to those chief phases of business and industrial life which are suitable for arithmetical treatment, and will summarize that important statistical data bearing upon agriculture, mining, and manufacture, the financial budgets of city, state, and nation, and the other large topics of public concern.